4.3: Angular Momentum

We wish to find out what it the meaning of "*l*" and "*m*" in the quantum numbers *(n,l,m)*.

Let us start with the classical formula for angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

Component by component in Cartesian coordinates this is:

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x.$$

To move into quantum mechanics we follow the usual recipe:

$$p_x \to -i\hbar\partial/\partial x, \; p_y \to -i\hbar\partial/\partial y, \; p_z \to -i\hbar\partial/\partial z$$

Do these operators commute? (HW12)

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= \underbrace{[yp_z, zp_x]}_{yp_x[p_z,z]} - \underbrace{[yp_z, xp_z]}_{yx\underbrace{[p_z,p_z]}_{0}} - \underbrace{[zp_y, zp_x]}_{p_y p_x\underbrace{[z,z]}_{0}} + \underbrace{[zp_y, xp_z]}_{xp_y[z,p_z]}$$

$$= yp_x\underbrace{[p_z, z]}_{-i\hbar} + xp_y\underbrace{[z, p_z]}_{+i\hbar} = i\hbar\underbrace{(xp_y - yp_x)}_{L_z} = i\hbar L_z$$

$$\boxed{[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y}$$

If operators do not commute, then we cannot know them simultaneously simply from the general theorem of Ch. 3. For example:

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2 = \frac{\hbar^2}{4}\langle L_z\rangle^2$$

However, something special happens with the square of the angular momentum:

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2$$

It commutes with L_x (and with L_y and with L_z):

$$[L^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$

… we need a “mini theorem” now …

$[L_y^2, L_x] = L_y^2 L_x - L_x L_y^2$

$L_y [L_y, L_x] = L_y (L_y L_x - L_x L_y)$

$[L_y, L_x] L_y = (L_y L_x - L_x L_y) L_y$

$[L_y^2, L_x] = L_y [L_y, L_x] + [L_y, L_x] L_y$ used in HW12

In general $[A^2, B] = A [A,B] + [A,B] A$. You will use this theorem in HW12. Applying this theorem multiple times:

$$
\begin{aligned}
[L^2, L_x] &= [L_x^2, L_x] + \underbrace{[L_y^2, L_x]} + \underbrace{[L_z^2, L_x]} \\
&= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z \\
&= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z \\
&= 0.
\end{aligned}
$$

The same holds for all components:

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0, \quad [L^2, L_z] = 0$$

Because L^2 commutes with at least one component (usually chosen to be L_z) then we should find **eigenstates of both operators simultaneously**.

$$L^2 f = \lambda f \qquad\qquad L_z f = \mu f$$

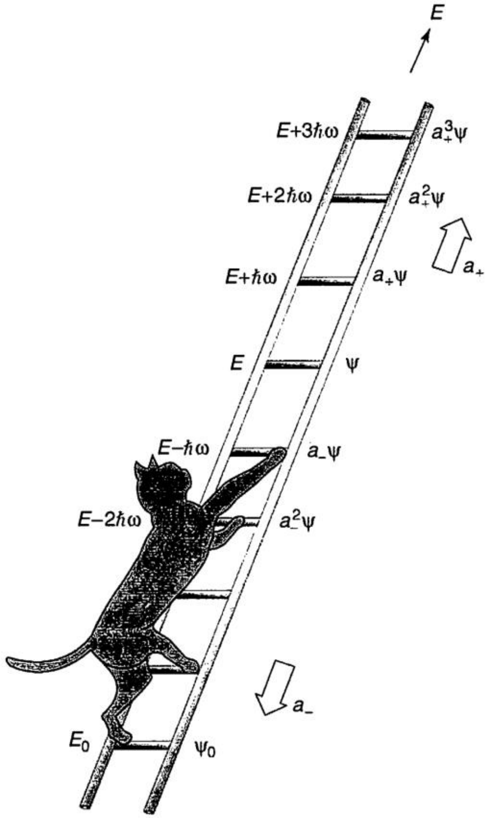
Then our mission is to find λ and μ (today) and f (next lecture).

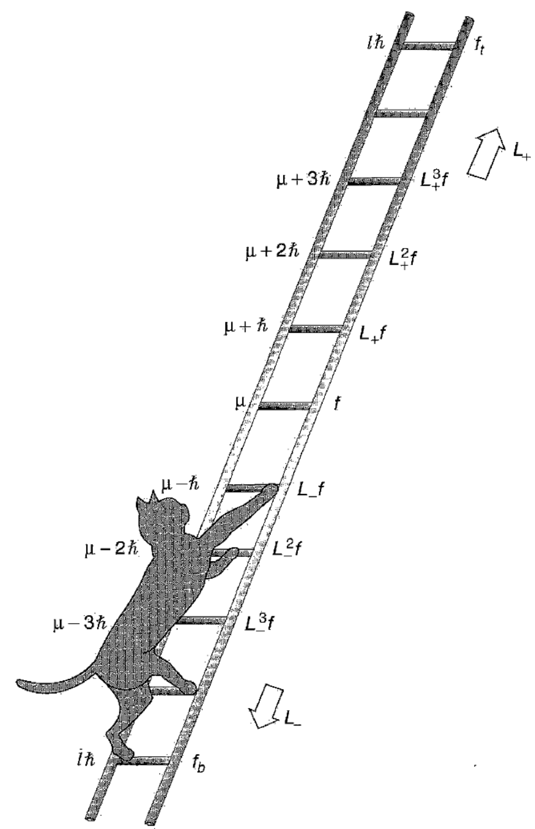
To solve this problem we will use a procedure very similar to that of the Harmonic Oscillator with the *lowering and raising operators.*

In Ch. 2 we defined the raising and lowering operators.

$$\boxed{a_{\pm} \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)}$$

$$\boxed{L_{\pm} \equiv L_x \pm iL_y}$$

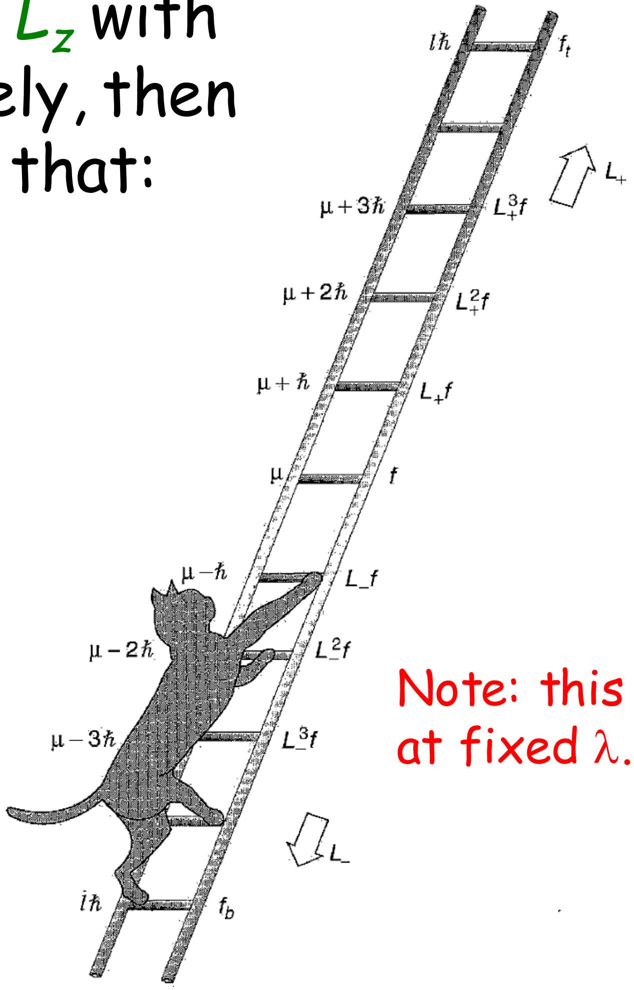




If f is eigenfunction of L^2 and L_z with eigenvalues λ and μ, respectively, then the claim (proof similar as in Ch2) is that:

$$\boxed{\begin{aligned} L^2(L_\pm f) &= \lambda(L_\pm f) \\ L_z(L_\pm f) &= (\mu \pm \hbar)(L_\pm f) \end{aligned}}$$

L_+ is the raising operator and L_- the lowering operator.



Note: this is at fixed λ.

But like in Ch.2 this cannot go on forever.
Eventually the projection, positive or negative, will be larger than the vector itself.

At the top value, let us call the L_z max eigenvalue $\hbar l$

$$L_z f_t = \hbar l f_t; \qquad L^2 f_t = \lambda f_t$$

Useful identity:

$$L_\pm L_\mp = (L_x \pm i L_y)(L_x \mp i L_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x)$$

$$= L^2 - L_z^2 \mp i(i\hbar L_z)$$ Or, just reorganizing:

$$\boxed{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$$

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t =$$

$$= (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar^2 l(l+1) f_t$$

$$\boxed{\lambda = \hbar^2 l(l+1)}$$

Too tedious to continue with all the details but you have the essence of the reasoning already. See pages 163-165.

$$\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m}$$

Analyzing now the bottom of the chain of states it can be shown that:

$$m = -l, \ -l+1, \ \ldots, \ l-1, \ l$$

thus $\quad l = 0, \ 1/2, \ 1, \ 3/2, \ \ldots$

Note that *l* can be integer or half-integer mathematically speaking … more later, maybe in last lecture (hint: spin).