In the previous lecture we found the meaning of "*l*" and "*m*" in the quantum numbers *(n,l,m)*.

The eigenvalues of the L^2 operator were $\hbar^2 l(l+1)$ and those of the L_z operator were $\hbar m$.

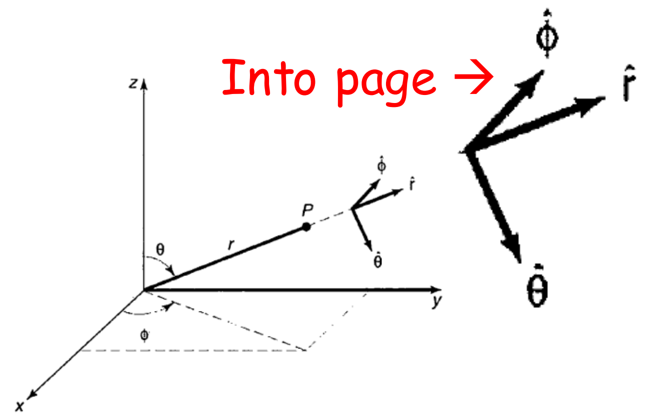
Now we need the eigenfunctions ...

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \mathbf{p} \to \frac{\hbar}{i}\nabla \qquad \mathbf{L} = (\hbar/i)(\mathbf{r} \times \nabla)$$

classical — classical to QM — QM angular momentum

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

(not obvious) gradient operator in spherical coordinates. Search in some math book.



Remember that $\mathbf{r} = r\hat{r}$. Then:

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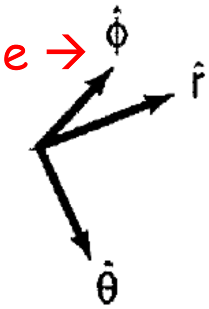

$$\mathbf{L} = \frac{\hbar}{i}\left[r\underbrace{(\hat{r}\times\hat{r})}_{=0}\frac{\partial}{\partial r} + \underbrace{(\hat{r}\times\hat{\theta})}_{=\hat{\phi}}\frac{\partial}{\partial\theta} + \underbrace{(\hat{r}\times\hat{\phi})}_{=-\hat{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

We arrive to $\mathbf{L} = \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$

We can now rewrite in terms of the unit vectors in Cartesian coordinates using:

$$\hat{\theta} = (\cos\theta\cos\phi)\hat{\imath} + (\cos\theta\sin\phi)\hat{\jmath} - (\sin\theta)\hat{k}$$

$$\hat{\phi} = -(\sin\phi)\hat{\imath} + (\cos\phi)\hat{\jmath},$$

By mere replacement (easy) we arrive to:

$$\boxed{L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)}$$

$$\boxed{L_y = \frac{\hbar}{i}\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)}$$

$$\boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$$

Why do we care so much about *x,y,z* components instead of θ,ϕ,r components? Because we have all commutators etc etc written in terms of x,y,z components from previous lecture …

The very important raising and lowering operators then become (again, easy):

$$L_{\pm} = L_x \pm i L_y = \pm \hbar e^{\pm i\phi} \left(\frac{\partial}{\partial\theta} \pm i \cot\theta \frac{\partial}{\partial\phi} \right)$$

Using a relation (easy) derived in the last lecture Nov. 20:

$$L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

we can deduce an expression (**not** easy, see my solution to problem 4.21 provided by email) for $\mathbf{L}^2$ in spherical coordinates:

$$\boxed{L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]}$$

So we finally arrived to the differential equation we wish to solve to find the eigenfunctions:

$$\boxed{L^2 f_l^m = \underbrace{-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}_{L^2\text{ from previous page}} f_l^m = \underbrace{\hbar^2 l(l+1)}_{\text{from previous lecture}} f_l^m}$$

HOWEVER, this equation happens to be the SAME "angular equation" that we derived at the start of Chapter 4 when we were trying "separation of variables" to solve the Sch. Eq. (see Eq.[4.17]):

The "angular equation" Eq.[4.17] was (a mere division by $-Y\,\hbar^2$ left and right is the only difference):

$$\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\}=-l(l+1)$$

We showed early in the chapter that the solution were the Spherical Harmonics, so there is no further work to do!

Same for the other (much easier) equation (left as exercise):

$$L_z f_l^m = \frac{\hbar}{i}\frac{\partial}{\partial\phi}f_l^m = \hbar m f_l^m$$

In summary: the Spherical Harmonics that we studied in detail before are the eigenfunctions of the $\mathbf{L}^2$ and L_z operators.

The eigenfunctions of the Hamiltonian of the Hydrogen atom

$$H\psi = E\psi, \quad L^2\psi = \hbar^2 l(l+1)\psi, \quad L_z\psi = \hbar m\psi$$

were already eigenfunctions of $\mathbf{L}^2$ and L_z

This completes the logic: the "l" and "m" quantum numbers introduced mathematically during the separation of variables procedure have a profound physical meaning.

$\boxed{\text{The eigenvalues of } L^2 \text{ are } \hbar^2 l(l+1) \text{ and those of } L_z \text{ are } \hbar m.}$

Note that there is a $l(l+1)$ not a l^2. At large "l" the difference is small but at say l=1 it is not small.