Moving $Y(\theta, \phi)\sin^2(\theta)$ let us focus on the Angular Equation which is the SAME for any potential V(r):

$$\boxed{\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2\theta\, Y}$$

Let us once again try separation of variables, followed by the canonical division by $\Theta\Phi$:

$$\boxed{Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)}$$

$$\underbrace{\left\{\frac{1}{\Theta}\left[\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\right] + l(l+1)\sin^2\theta\right\}}_{\text{function of } \theta \text{ only}} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{function of } \phi \text{ only}} = 0$$

Note total derivative used instead of partial only because unknown function depends only on $\theta$

Like in several occasions before, then this means:

$$\boxed{\frac{1}{\Theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]+l(l+1)\sin^2\theta=m^2}$$

$$\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2}$$

Thus, up to now via the trick of the separation of variables we have managed to write the single monstrous Sch. Eq. in spherical coordinates, into three separated equations.

In doing so, we introduced three unknown constants E (energy of the stationary states that likely will have an index *n* for bound states), plus l(l+1) and m.

Thus, solutions will have THREE labels: n,l,m

At least we have ONE break. The equation for $\phi$ is easy!

$$\boxed{\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \;\Rightarrow\; \Phi(\phi) = e^{im\phi}}$$

Because "physics" has to be the same after a $2\pi$ rotation then we impose:

$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

which leads to quantization of m (discrete values):

$$\boxed{m = 0, \pm 1, \pm 2, \ldots}$$

The second angular equation is NOT easy:

$$\boxed{\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + [l(l+1)\sin^2\theta - m^2]\Theta = 0}$$

The solution is simply given to you:

$$\boxed{\Theta(\theta) = A\,\underbrace{P_l^m(\cos\theta)}_{}}$$

Called the **associated Legendre function** (not polynomials)

For m=0, they are just called the **Legendre polynomials** (yes, now polynomials).

These m=0 polynomials will be given to you. At most you will be asked to prove that indeed they are solutions. Using the common notation *cos*($\theta$)= *x* :

$$P_0 = 1$$

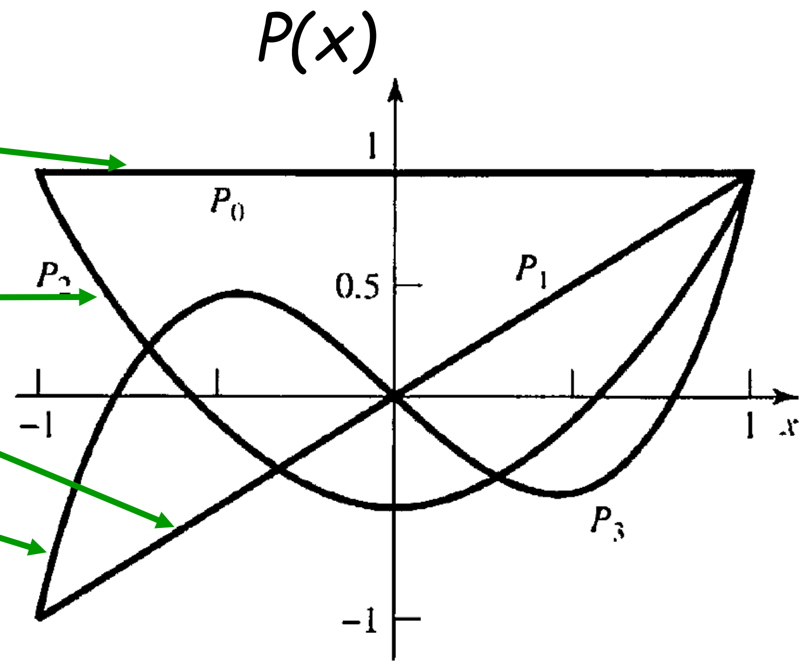
$$P_1 = x$$

$$P_2 = \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$




These polynomials can be divided in even and odd.

But we need the entire associated Legendre functions!
Just accept the following formula:

$$\boxed{P_l^m(x) \equiv (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_l(x)}$$

$$\underbrace{P_2^0(x) = \frac{1}{2}(3x^2-1),}_{l=2,\ m=0} \quad \underbrace{P_2^1(x) = (1-x^2)^{1/2}\frac{d}{dx}\left[\frac{1}{2}(3x^2-1)\right] = 3x\sqrt{1-x^2},}_{l=2,\ m=1 \text{ or } -1}$$

$$\underbrace{P_2^2(x) = (1-x^2)\left(\frac{d}{dx}\right)^2\left[\frac{1}{2}(3x^2-1)\right] = 3(1-x^2),}_{l=2,\ m=2 \text{ or } -2}$$

Not a polynomial.
But note $(1-x^2)$ is
simply $\sin^2(\theta)$!

Note that in the formula of previous page "*l*" is the order of the polynomial, thus if *|m|>l*, I get a zero.

$$\boxed{P_l^m(x) \equiv (1-x^2)^{|m|/2} \left(\frac{d}{dx}\right)^{|m|} P_l(x)}$$

As a consequence we have the following constraints:

$$\boxed{l = 0,\ 1,\ 2, \ldots; \quad m = -l,\ -l+1,\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ l-1,\ l}$$

For each integer "l", there are (2l+1) values of m.

All associated Legendre functions you may need will be given to you in exams (in HW you may have to look for them in tables or googling):

| | |
|---|---|
| $P_0^0 = 1$ | $P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$ |
| $P_1^1 = \sin\theta$ | $P_3^3 = 15\sin\theta(1-\cos^2\theta)$ |
| $P_1^0 = \cos\theta$ | $P_3^2 = 15\sin^2\theta\cos\theta$ |
| $P_2^2 = 3\sin^2\theta$ | $P_3^1 = \frac{3}{2}\sin\theta(5\cos^2\theta - 1)$ |
| $P_2^1 = 3\sin\theta\cos\theta$ | $P_3^0 = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$ |

If you start plotting these functions, the “orbitals” that you have seen many times before since high school start appearing!

“s” orbital

“p” orbital

“d” orbital