We will do the even sector while the odd will be in HW. Only F and D are unknowns.

$$\psi(x) = \begin{cases} Fe^{-\kappa x}, & \text{for } x > a, \\ D\cos(lx), & \text{for } -a < x < a, \\ \psi(-x), & \text{for } x < -a \end{cases}$$

even

Continuity of ψ at x=a: $$Fe^{-\kappa a} = D\cos(la)$$

Continuity of dψ/dx at x=a: $$-\kappa Fe^{-\kappa a} = -lD\sin(la)$$

From ratio we get $\boxed{\kappa = l\tan(la)}$ where

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar} \quad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

This equation cannot be solved exactly. Must be done numerically. *In general this is the most common situation.*

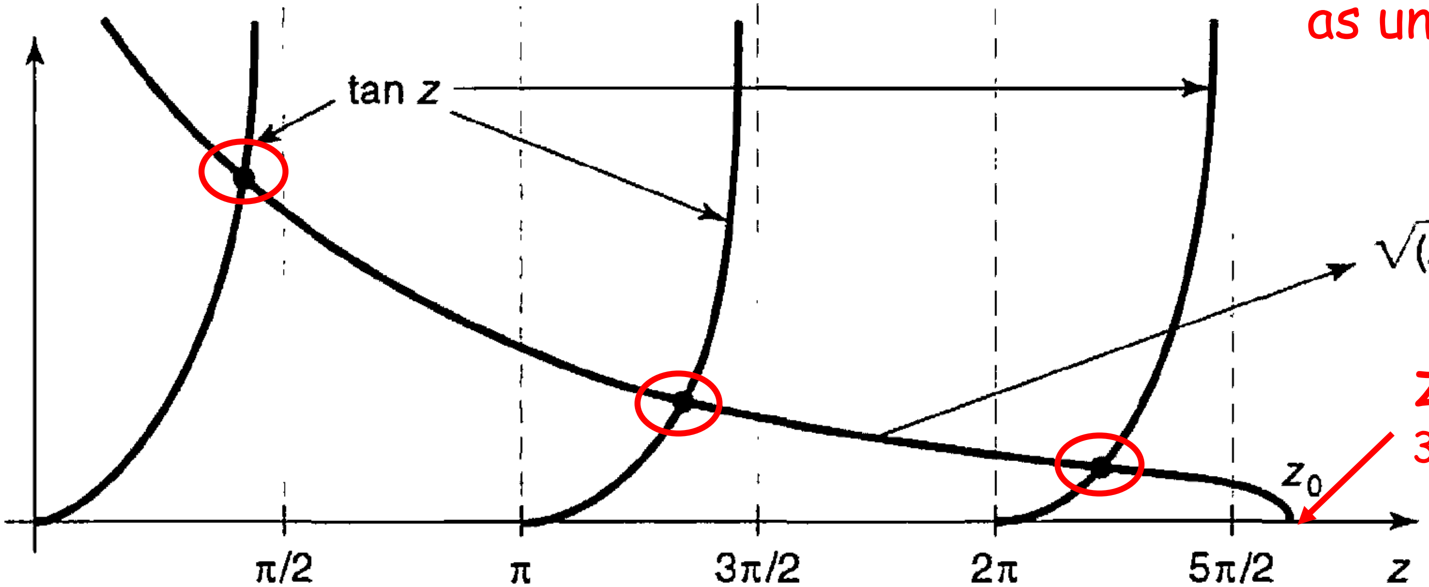
Also in general there is no need to use all the numerical values of masses, Planck constant, etc. Use clever variables.

$$z \equiv la \qquad z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0} \quad \rightarrow \quad \kappa a = \sqrt{z_0^2 - z^2}$$

Dimensionless combo

$$\rightarrow \quad \tan z = \sqrt{(z_0/z)^2 - 1}$$

We trade E as unknown to z as unknown.

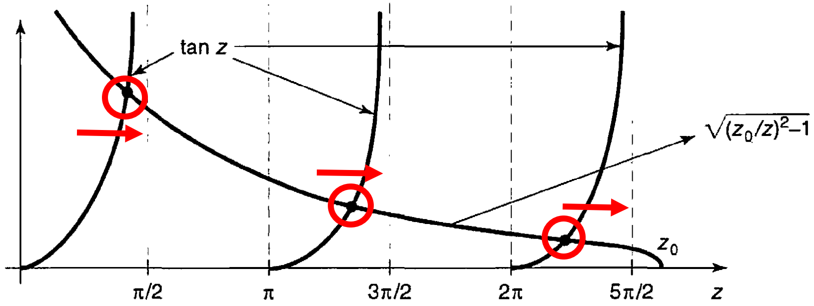


$z_0=8$

3 solutions

For each value of $z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$ there will be a different finite number of solutions. Consider two special limits:

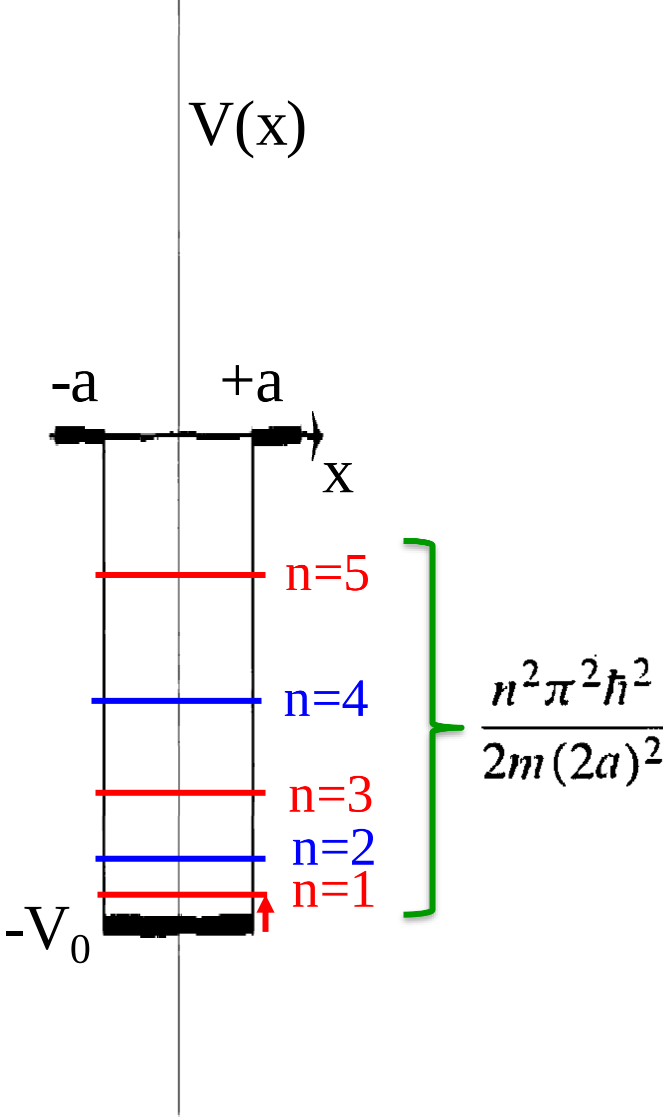
(1) V_0 large i.e. a **very deep well**. This means z_0 large.



→ solutions are (n odd) $z_n = n\pi/2$

Because $z \equiv la$, then $z_n = n\pi/2$ means $l_n = n\pi/2a$, and

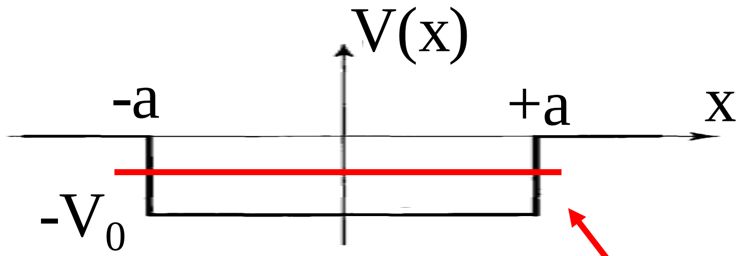
$$l_n \equiv \frac{\sqrt{2m(E_n + V_0)}}{\hbar} \quad \text{thus} \quad \boxed{E_n + V_0 \cong \frac{n^2\pi^2\hbar^2}{2m(2a)^2}}$$

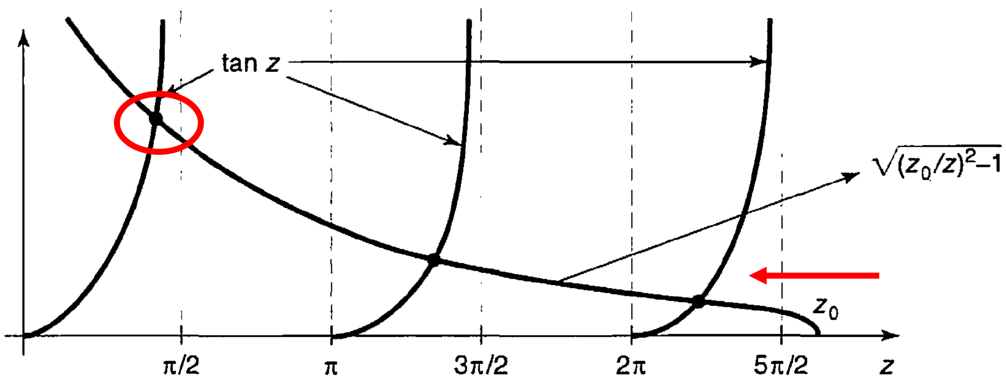


These are "half" the solutions ("even" under x→-x, i.e. n odd) we found before for the infinite square well assuming width "2a".

For any large but finite V_0 the number of solutions will be large but finite as well.

(2) V_0 small i.e. a **very shallow well**. This means z_0 small.



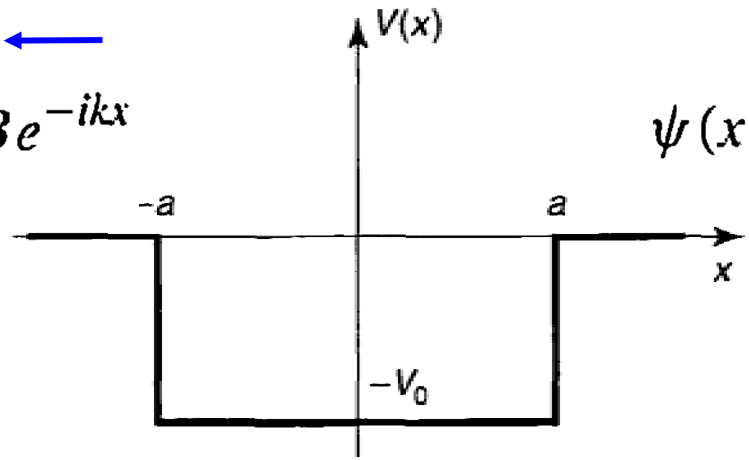


As z_0 is reduced, the number of solutions decreases but always one survives no matter how weak the potential is!

Consider now the scattering states E>0 (unrestricted).

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$
left

$$k \equiv \frac{\sqrt{2mE}}{\hbar}$$



$$\psi(x) = Fe^{ikx}$$
right

$$\psi(x) = C\sin(lx) + D\cos(lx) \qquad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$
inside

Continuity of ψ and dψ/dx at x=-a:

$$\begin{cases} Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la) \\ ik[Ae^{-ika} - Be^{ika}] = l[C\cos(la) + D\sin(la)] \end{cases}$$

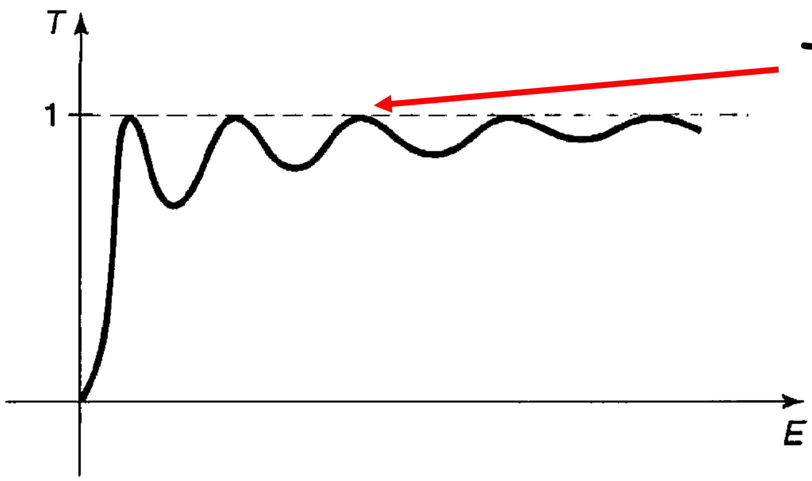
Continuity of ψ and dψ/dx at x=+a:

$$\begin{cases} C\sin(la) + D\cos(la) = Fe^{ika} \\ l[C\cos(la) - D\sin(la)] = ikFe^{ika} \end{cases}$$

This system of 4 equations and 5 unknowns can be solved as for the delta potential via ratios.

Recall $T = |F|^2/|A|^2$ is the transmission coefficient

The result is
$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$



T=1 if $\frac{2a}{\hbar}\sqrt{2m(E_n+V_0)} = n\pi$

Thus, for some particular energies there is perfect transmission !! And they happen to be the solutions of the infinite square well. Weird QM!

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

There are like "ghosts" of the infinite well ...