For example, in the infinite square well the set of functions that are normalizable i.e.

$$\int_0^a |f(x)|^2\,dx < \infty,$$

define the "Hilbert space of the infinite square well". In general, in the [a,b] interval:

$$\underbrace{\int_a^b |f(x)|^2\,dx < \infty}_{\text{We say the function is "square integrable"}} \qquad \underbrace{\langle f|g\rangle \equiv \int_a^b f(x)^* g(x)\,dx}_{\text{This is the "inner product" of f(x) with g(x) (like dot product in 3D)}}$$

Properties of inner products:

$$\langle f|g\rangle^* = \left[\int_a^b f(x)^* g(x)\,dx\right]^* = \int_a^b g(x)^* f(x)\,dx = \langle g|f\rangle$$

Then: $\boxed{\langle g|f\rangle = \langle f|g\rangle^*}$

Prof recommendation: If in doubt, work with the explicit integrals as definition of inner product !

In particular: $\langle f|f\rangle = \int_a^b |f(x)|^2\,dx$

It is real and non negative, justifying the conjugation in f(x) in the definition of <f|g>. <f|f>=1 means normalized to 1.

In this notation, then "orthonormality" of functions is written as:

$$\boxed{\langle f_m | f_n \rangle = \delta_{mn}}$$

"Completeness": any function in the Hilbert space of the problem can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$|f\rangle = \sum_n c_n |f_n\rangle$$

$$\langle f_m | f\rangle = \sum_n c_n \underbrace{\langle f_m | f_n\rangle}_{\delta_{mn}}$$

Then, the coefficients are $\boxed{c_n = \langle f_n | f \rangle}$ which is what we knew:

$$c_n \equiv \int_a^b f_n(x)^* f(x)\, dx$$

One last observation: the Schwarz inequality

In three dimensions, it is obvious:

$$(\mathbf{a.b})^2 = | \mathbf{a} |^2 \, | \mathbf{b} |^2 \cos\phi \leq | \mathbf{a} |^2 \, | \mathbf{b} |^2$$

In N dimensions, it can be shown that:

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle$$

or, more explicitly, and taking square roots:

$$\boxed{\left|\int_a^b f(x)^* g(x)\,dx\right| \leq \sqrt{\int_a^b |f(x)|^2\,dx \int_a^b |g(x)|^2\,dx}.}$$

Observables and Hermitian Operators

Expectation values of operators, such as $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{H}\rangle$, … can be expressed in general as:

$$\boxed{\langle\hat{Q}\rangle = \int \Psi^* \hat{Q} \Psi \, dx = \langle\Psi|\hat{Q}\Psi\rangle}$$

where $\hat{Q}$ is the $\hat{x}$, $\hat{p}$, $\hat{H}$, … operator.

But if $\hat{Q}$ is related with a measurement, physically we expect to obtain real numbers. Thus, for the operators of relevance:

$$\langle\hat{Q}\rangle = \langle\hat{Q}\rangle^*$$

Since $\langle \hat{Q} \rangle = \langle \hat{Q} \rangle^*$ means

$$\langle \Psi | \hat{Q} \Psi \rangle = \langle \Psi | \hat{Q} \Psi \rangle^*$$ <- from physics common sense

and since $\langle f | g \rangle^* = \langle g | f \rangle$ <- from math

($g = \hat{Q}\Psi$, $f = \Psi$)

then $\boxed{\langle \Psi | \hat{Q} \Psi \rangle = \langle \hat{Q} \Psi | \Psi \rangle}$

Operators $\hat{Q}$ that satisfy this property are called **Hermitian operators**.

Thus, observables are represented by Hermitian operators

Example 1: easy … coordinate $\hat{x}$ operator

$$\langle \hat{x} \rangle = \int \underbrace{\Psi^* \hat{x} \Psi \, dx}_{\langle \Psi | \hat{x} \Psi \rangle} = \int (x\Psi)^* \Psi \, dx = \langle \hat{x}\Psi | \Psi \rangle$$

$$\boxed{\langle \Psi | \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi | \Psi \rangle}$$

Thus, $\hat{x}$ is Hermitian

$(\hat{x})^2$ is also Hermitian, and in general potentials $V(\hat{x})$, like harmonic oscillator, are Hermitian.

Example 2: difficult … momentum $\hat{p}$ operator

$$\langle \hat{p} \rangle = \int \underbrace{\Psi^* \hat{p} \Psi \, dx}_{\langle \Psi | \hat{p} \Psi \rangle} = -i\hbar \int \Psi^* \frac{d}{dx} \Psi \, dx$$

$$\boxed{\int_{-\infty}^{\infty} \Psi^* \frac{d}{dx} \Psi \, dx = \underbrace{\Psi^* \Psi \Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \left(\frac{d}{dx} \Psi^*\right) \Psi \, dx}$$

Thus, so far $\boxed{\langle \Psi | \hat{p} \Psi \rangle} = -i\hbar \int \Psi^* \frac{d}{dx} \Psi \, dx$

$$= (-i\hbar)\left(- \int_{-\infty}^{\infty} \left(\frac{d}{dx} \Psi^*\right) \Psi \, dx \right) = \int_{-\infty}^{\infty} \left(-i\hbar \frac{d}{dx} \Psi \right)^* \Psi \, dx = \boxed{\langle \hat{p} \Psi | \Psi \rangle}$$

Because $\langle \Psi | \hat{p}\, \Psi \rangle = \langle \hat{p} \Psi | \Psi \rangle$

then, $\hat{p}$ is a "Hermitian operator"

$-i \frac{d}{dx}$ is Hermitian ($\hbar$ is just a constant)

$- \frac{d}{dx}$ is not Hermitian

$(-i \frac{d}{dx})^2$ is Hermitian

$\hat{H} = \hat{p}^2/2m + V(\hat{x})$ is Hermitian