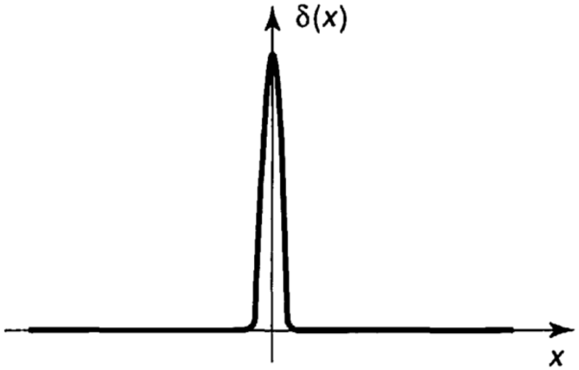
The delta function potential

The Dirac delta function represents a localized heavy object like a neutron that electrons may collide with.



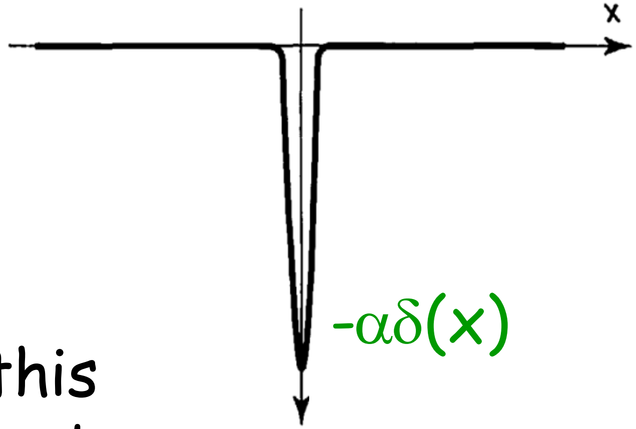
$$\delta(x) \equiv \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}. \quad \text{with} \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\, dx = f(a) \int_{-\infty}^{+\infty} \delta(x-a)\, dx = f(a)$$

The Dirac delta function can be positive (repulsive, as shown) or negative (attractive).

Consider a potential $V(x) = -\alpha\delta(x)$. The Sch. Eq. is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi$$



Because potential at +- infinity is 0, this potential can have both bound (E<0) and scattering (E>0) states.

If x is nonzero, and E<0 to explore bound states, the Sch. Eq. is "almost" the same as for free particle:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

Sign difference with free particle

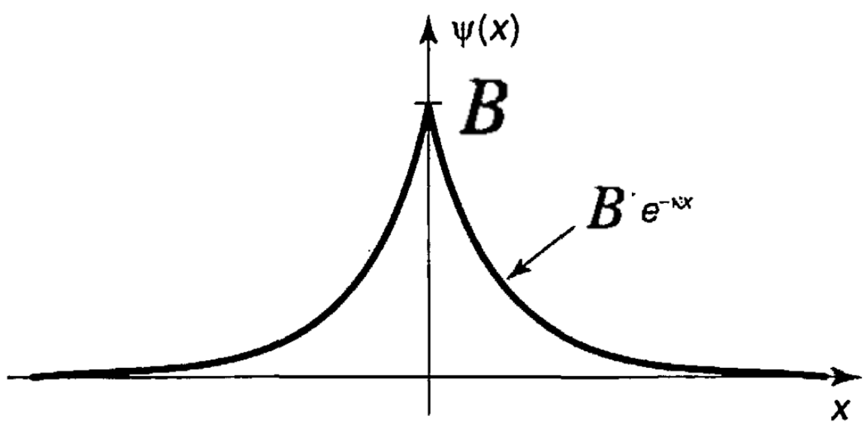
Because $\kappa^2>0$, then look for solutions:

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$$

For x<0, A=0 otherwise diverges. $\psi(x) = Be^{\kappa x}$ $(x < 0)$

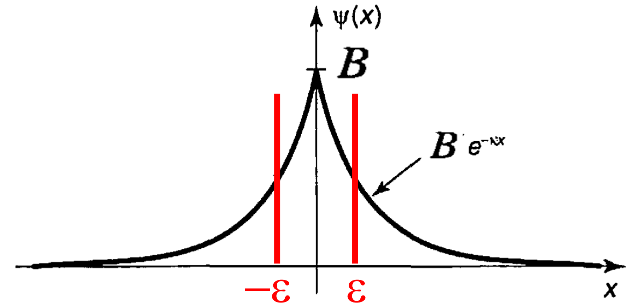
For x>0, similar reasoning leads to: $\psi(x) = Fe^{-\kappa x}$ $(x > 0)$

By continuity at x=0, then B=F.



We have the shape of Ψ but not E yet, plus we have not used α.

How do we find E? In dealing with δ-functions, often we need to integrate near the δ-function. Use this "trick" for Sch. Eq., send epsilon -> 0 at the end.



$$\underbrace{-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx}_{\underbrace{-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{+\epsilon}-\left.\frac{d\psi}{dx}\right|_{-\epsilon}\right)}_{\text{Easy, since I have }\Psi\text{ already.}}} + \underbrace{\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx}_{-\alpha\,\underbrace{\psi(0)}_{B\,(=F)}} = \underbrace{E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx}_{0\text{ because }\Psi\text{ continuous}}$$

$$\begin{cases} d\psi/dx = -B\kappa e^{-\kappa x}, & \text{for } (x > 0), \quad \text{so } d\psi/dx\big|_{+} = -B\kappa, \\ d\psi/dx = +B\kappa e^{+\kappa x}, & \text{for } (x < 0), \quad \text{so } d\psi/dx\big|_{-} = +B\kappa. \end{cases}$$

Then, integrated Sch. Eq. becomes $-\frac{\hbar^2}{2m}(-2\cancel{B}\kappa) - \alpha\,\cancel{B} = 0$

$$\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

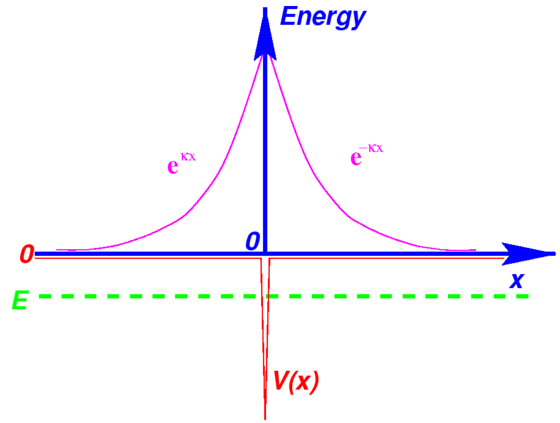
Note: *B* cancels and ε does not appear.

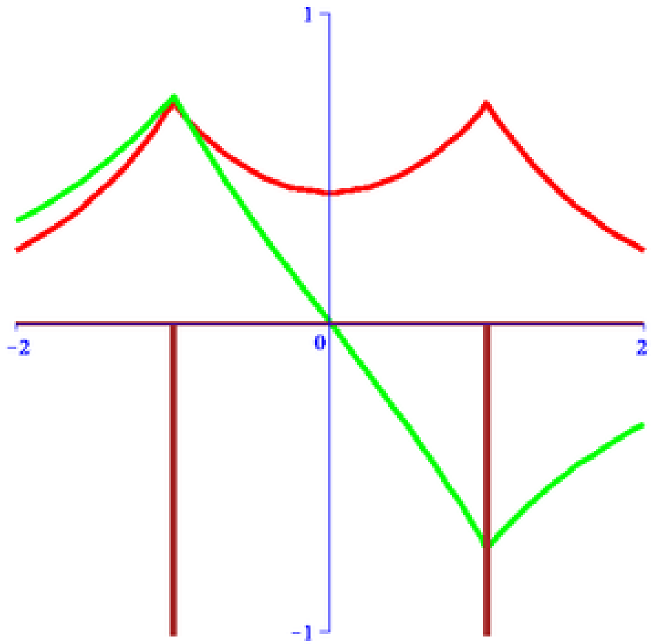
$$\boxed{E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}}$$

(1) *B* arises from normalization, left as exercise.
(2) Only because *V(x)* diverges, $d\Psi/dx$ is discontinuous.
Otherwise it is continuous (i.e. second term, previous page, gives 0)

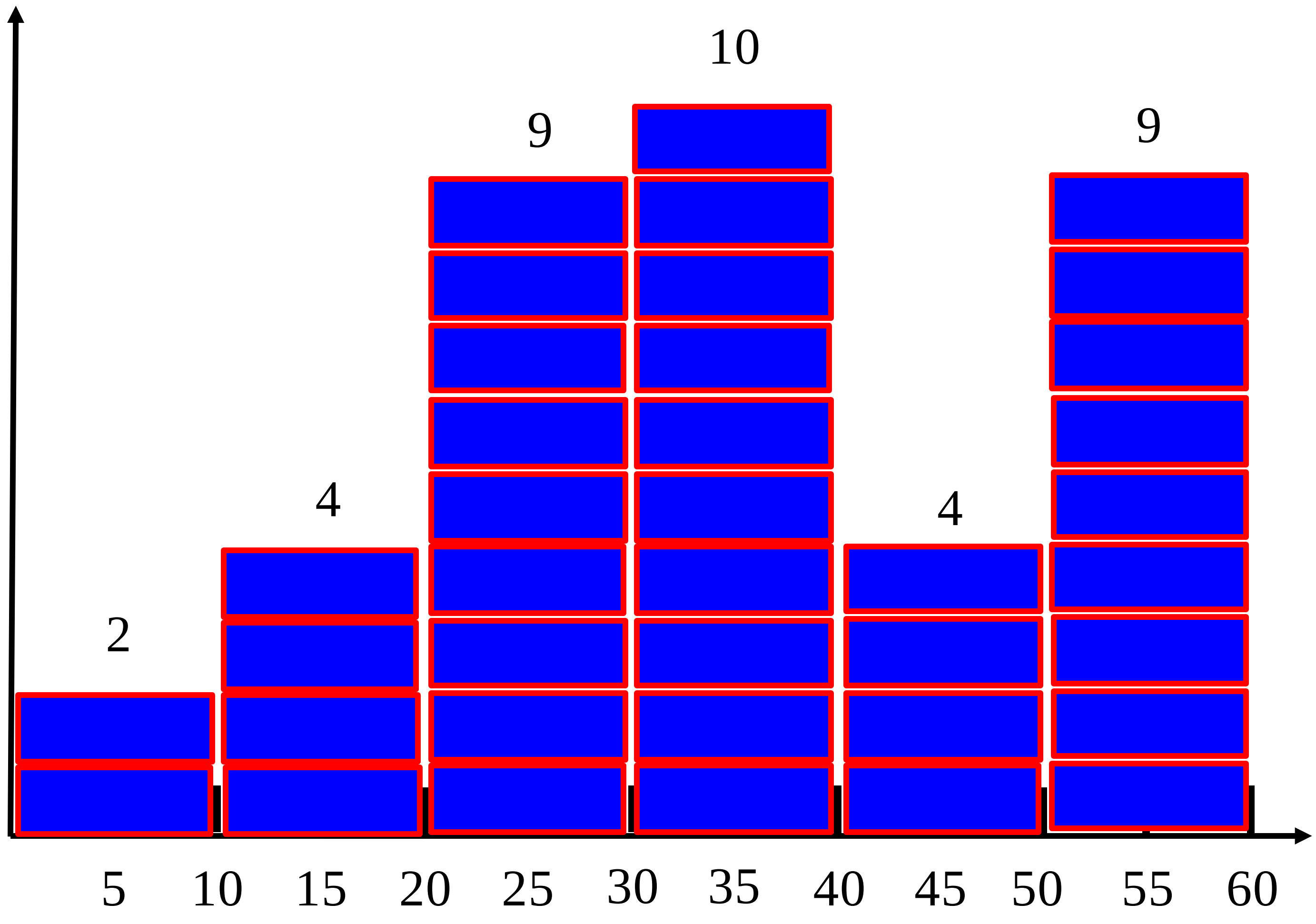
The (only) bound state and its energy s:

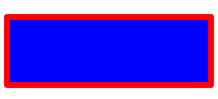
$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \qquad E = -\frac{m\alpha^2}{2\hbar^2}.}$$





NOTE: Generalization to two deltas will have two solutions (even and odd) with different energies.

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Results Test 1 Sept 27 2018

Average: 35

Comments about Test 1:

It was easier than you thought. The first problem had a constant wave function. Cannot be simpler.

A problem like $[x^2,p]$ was done in class and in book (page 43) for the case $[x,p]$. You cannot use the a+ a- operators here.

The problem with the time dependence was given in a form where you only needed to add exp{-i En t / hbar} to each Ψn.

The problem with $\langle p^2 \rangle$ using a+ and a-: $\langle x^2 \rangle$ was done in class and the book (page 49). Just one sign difference.

The problem with the Gaussian integral, you had the integral in the PDF with formulas. Only integrals you had to know were that of 1, x, x^2, and sin(ax).

My main recommendation:

The way to prepare for exams is to follow “line by line, equal sign by equal sign” the lectures and the homework solutions, without fooling yourself.

This is NOT casual reading. Every detail, every step matters.