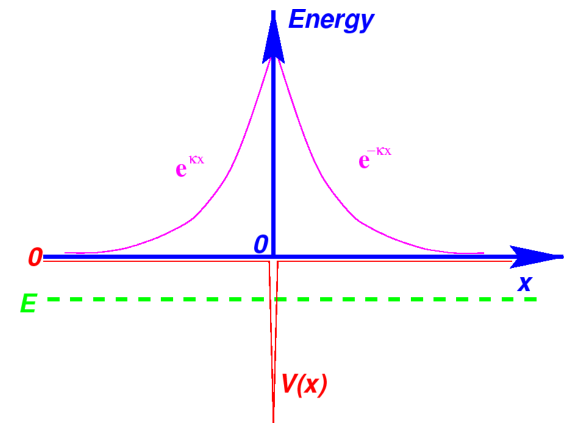
Example 3.4 book:

Consider a particle located in the (only) bound state of the δ-function potential. The wave function is:

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} e^{-iEt/\hbar}$$



Typical question: what is the probability of measuring a momentum greater than $p_0 = m\alpha/\hbar$*? We need to calculate $|c(p)|^2$ (see next page).

* Can you confirm that the units are those of momentum?

Reminder of previous lecture: eigenfunction of momentum operator $\hat{p}$.

We need an eigenfunction $f_p(x)$ such that

$$\frac{\hbar}{i}\frac{d}{dx}f_p(x) = p f_p(x)$$

$\hat{p}$ operator

eigenfunction

eigenvalue

Solution is very easy (but normalization is complicated):

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} e^{-iEt/\hbar}$$

Repeating: What is the probability of measuring a momentum greater than $p_0 = m\alpha/\hbar$?

We need to calculate $|c(p)|^2$.

$$c(p) = \langle f_p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx =$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{m\alpha}}{\hbar} e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-m\alpha|x|/\hbar^2}\, dx = \boxed{\sqrt{\frac{2}{\pi}} \frac{p_0^{3/2} e^{-iEt/\hbar}}{p^2 + p_0^2}}$$

integral given

Integral given. Answer must be a number [0,1]

The final answer is:

$$\int_{p_0}^{\infty} |c(p)|^2 dp = \frac{2}{\pi} p_0^3 \int_{p_0}^{\infty} \frac{1}{(p^2 + p_0^2)^2} dp = 0.0908$$

Just a name: c(p), which can be function of t but not x, is often called Φ(p,t), the momentum space wave function.

A similar problem could be formulated for the harmonic oscillator involving Gaussians (problem 3.11, HW8) or the infinite square well involving sines, etc., etc.

Uncertainty Principle *(4 pages, prepare for impact)*

The standard deviation for any operator is

$$\sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 = \langle \Psi | \hat{A}^2 | \Psi \rangle - \langle \Psi | \langle \hat{A} \rangle \hat{A} | \Psi \rangle =$$

$$= \langle \Psi | \hat{A}^2 - 2 \langle \hat{A} \rangle \hat{A} + \langle \hat{A} \rangle \hat{A} | \Psi \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \Psi \rangle$$

If Ψ normalized to 1
i.e. $< \Psi | \Psi > = 1$.

For Hermitian operator this can be rewritten as:

For $\hat{A}$: $\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)\Psi | (\hat{A} - \langle \hat{A} \rangle)\Psi \rangle = \langle f | f \rangle$ where

$$f \equiv (\hat{A} - \langle \hat{A} \rangle)\Psi$$

For $\hat{B}$: $\sigma_B^2 = \langle g | g \rangle$ $\qquad g \equiv (\hat{B} - \langle \hat{B} \rangle)\Psi$

Consider the Schwartz inequality:

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

z

We use now the following property of complex numbers:

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 \geq [\mathrm{Im}(z)]^2 = \left[\frac{1}{2i}(z - z^*) \right]^2$$

Consider $z = \langle f | g \rangle$

Then .... $$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [\langle f | g \rangle - \langle g | f \rangle] \right)^2$$

Then (repeated) .... $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [\langle f | g \rangle - \langle g | f \rangle] \right)^2$

$\hat{A}$ is Hermitian

$$\langle f | g \rangle = \langle (\hat{A} - \langle \hat{A} \rangle)\Psi | (\hat{B} - \langle \hat{B} \rangle)\Psi \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle)\Psi \rangle$$
$$= \langle \Psi | (\hat{A}\hat{B} - \hat{A}\langle \hat{B} \rangle - \hat{B}\langle \hat{A} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle)\Psi \rangle$$
$$= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle \hat{B} \rangle \langle \Psi | \hat{A}\Psi \rangle - \langle \hat{A} \rangle \langle \Psi | \hat{B}\Psi \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle \langle \Psi | \Psi \rangle$$
$$= \langle \hat{A}\hat{B} \rangle - \langle \hat{B} \rangle \langle \hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle$$
$$= \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle.$$

Make sure you understand every step

$\langle g | f \rangle = \langle \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$ Left as exercise

In summary: $\langle f | g \rangle - \langle g | f \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle = \langle [\hat{A}, \hat{B}] \rangle$

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Generalized uncertainty principle:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Assumes <..> is in a normalized to 1 state, and both operators Hermitian.

As special case, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, then $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2 \qquad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

This was the last item of Ch. 3 for us