Consider now scattering states with $E>0$. If x nonzero, then the Sch. Eq. is the same as for free particles.

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad x<0$$

$$\psi(x) = Fe^{ikx} + Ge^{-ikx} \quad x>0$$

Continuity at x=0:

$$\boxed{F + G = A + B}$$

$$d\psi/dx = ik\left(Fe^{ikx} - Ge^{-ikx}\right) \quad x>0$$

$$d\psi/dx = ik\left(Ae^{ikx} - Be^{-ikx}\right) \quad x<0$$

Remember here the derivative is discontinuous at x=0.

From the same bound state analysis done before we find:

$$-\frac{\hbar^2}{2m}\underbrace{\left(\frac{d\psi}{dx}\bigg|_{+\epsilon}-\frac{d\psi}{dx}\bigg|_{-\epsilon}\right)}_{ik(F-G-A+B)} - \alpha\,\underbrace{\psi(0)}_{A+B} = 0$$

$$\boxed{ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)}$$

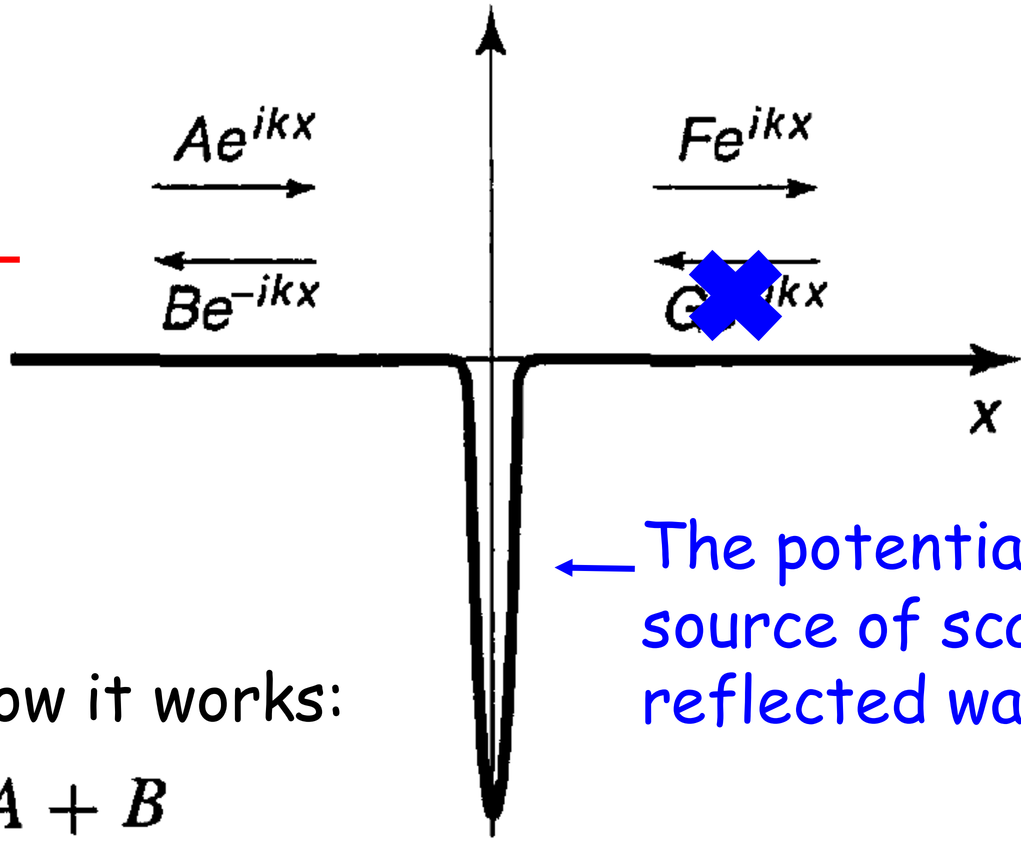
Four unknowns and two equations. Something missing …

These are not normalizable states so "strange" behavior is expected. We need to think "physically" what we are doing in terms of a real scattering experiment.

Real scattering experiment:

Source →

Detector of reflected e- ←



→ Detector of transmitted e-

The potential at x=0 is a source of scattering, and reflected waves.

Example of how it works:

$$F + \cancel{G} = A + B$$

Divide by A both eqs:

$$F/A = 1 + B/A$$

Now only two unknowns!

$$ik(F - \cancel{G} - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B)$$

Divide by A and again only B/A and F/A are unknowns.

Introducing $\beta \equiv \dfrac{m\alpha}{\hbar^2 k}$ it can be shown that:

$$\boxed{B = \frac{i\beta}{1 - i\beta}A, \quad F = \frac{1}{1 - i\beta}A}$$

Because probabilities are related to ψ^2 what matters are:

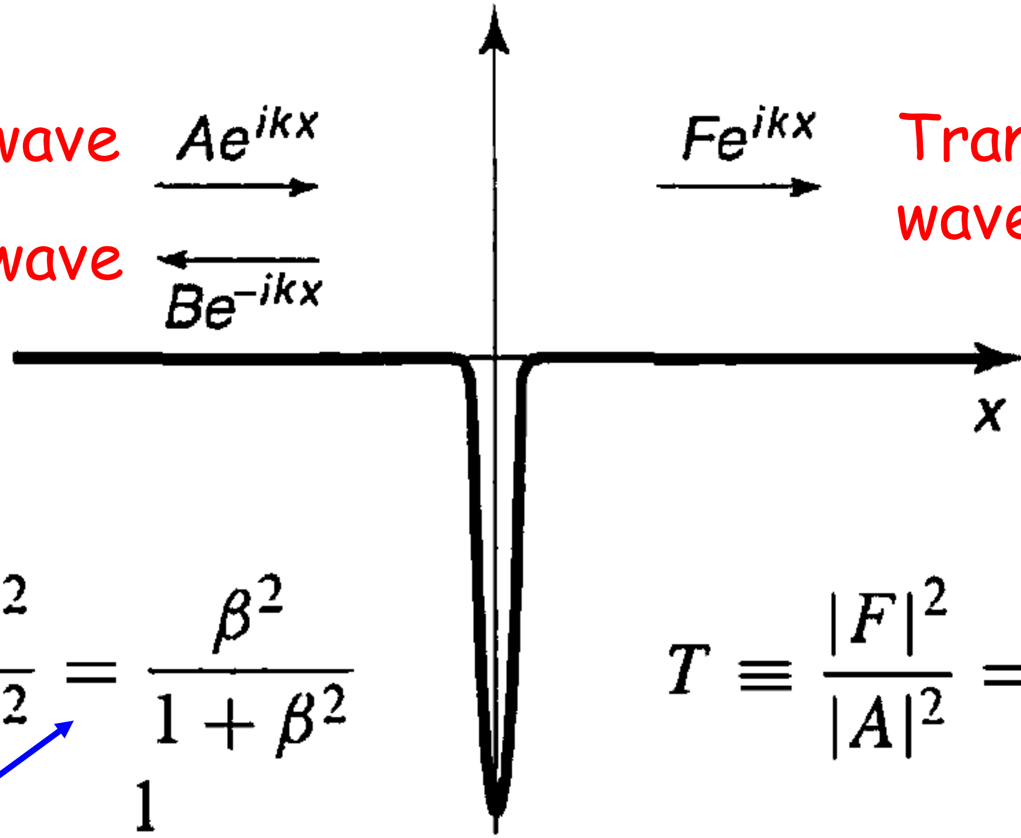
$$R \equiv \frac{|B|^2}{|A|^2} \qquad T \equiv \frac{|F|^2}{|A|^2}$$

They should satisfy: $R + T = 1$

Summary scattering experiment (any E>0):

Incident wave

Reflected wave



Transmitted wave

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} = \frac{1}{1+(2\hbar^2 E/m\alpha^2)}$$

δ potential

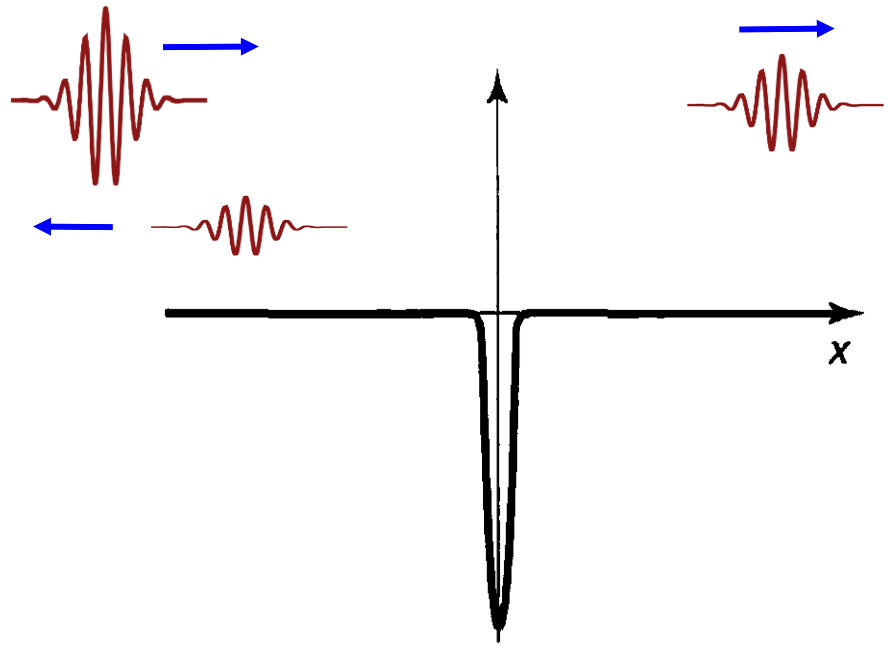
$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} = \frac{1}{1+(m\alpha^2/2\hbar^2 E)}$$

δ potential

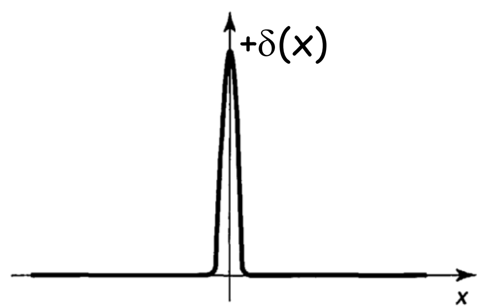
$$R + T = 1 \checkmark$$

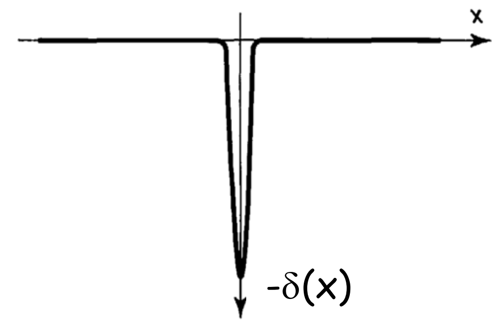
Two interesting comments:

(1) We used not normalizable solutions, but we meant to use **wave packets**:

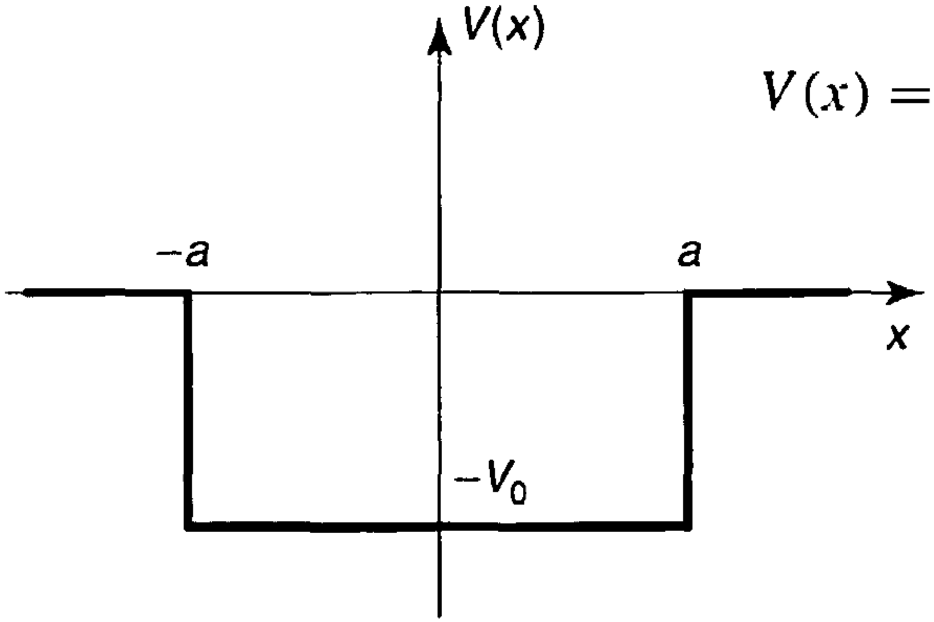


(2) For the scattering problem the sign of α does not matter !!
Repulsive or attractive is the same (but not for bound state).





The finite square well



$$V(x) = \begin{cases} -V_0, & \text{for } -a \le x \le a, \\ 0, & \text{for } |x| > a, \end{cases}$$

We anticipate it will have both bound and scattering states.

PROCEDURE: There are three regions. We will propose a general solution in each, and then match ψ and $d\psi/dx$ at the two boundaries.

Let us start with bound states i.e. E<0.

Left region x<-a and **Right** region x>a:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \kappa^2\psi \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$\left.\begin{aligned} \psi(x) &= Be^{\kappa x}, \quad \text{for } x < -a. \\ \psi(x) &= Fe^{-\kappa x}. \quad \text{for } x > a. \end{aligned}\right\}$$

The "other" exponential diverges in each case.

Middle region -a<x<a. Here E>-V_0

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -l^2\psi$$

$$l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar} > 0$$

In middle region, the general solution is:

$$\psi(x) = C\sin(lx) + D\cos(lx)$$

I cannot drop any term a priori.

Note: I could have proposed a sum of e^{ilx} and e^{-ilx} with two unknowns as well.

Five unknowns *A,C,D,F* (plus *E*) but I have five eqs: match of ψ and $d\psi/dx$ at x=a and x=-a, and normalization.

Moreover, the solutions must be even or odd under x → -x. *I can study each sector separately.*

We will do the even sector while the odd will be in HW. Only F and D are unknowns.

even

$$\psi(x) = \begin{cases} Fe^{-\kappa x}, & \text{for } x > a, \\ D\cos(lx), & \text{for } -a < x < a, \\ \psi(-x), & \text{for } x < -a \end{cases}$$

Continuity of ψ at x=a: $$Fe^{-\kappa a} = D\cos(la)$$

Continuity of $d\psi/dx$ at x=a: $$-\kappa Fe^{-\kappa a} = -lD\sin(la)$$