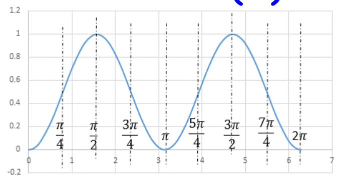
To finish the problem neatly, find *A* such that the wave function is normalized to 1.

result n independent

k=nπ/a, use u=kx

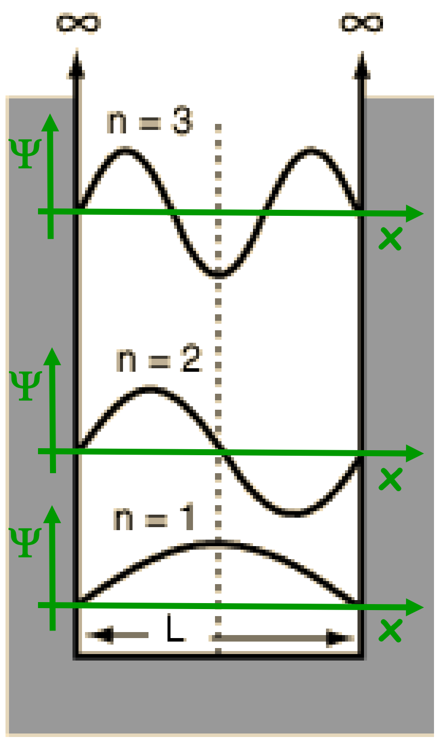
$$\int_0^a |A|^2 \sin^2(kx)\,dx = |A|^2 \frac{a}{2} = 1$$

Y=sin²(x)



The final complete solution then is:

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \quad , \quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$



x = 0 at left wall of box.

Only valid for V(x)=V(-x) potentials

2nd excited state	even	2 nodes
1st excited state	odd	1 node
ground state	even	0 node

Remember there is a $e^{-iE_n t/\hbar}$ multiplying always.

Thus, Re Ψ and Im Ψ parts are oscillating with time. But $|\Psi|^2$ is time independent.

Two neat properties of the solutions.
(I) They are orthonormal.

$$\int \psi_m(x)^* \psi_n(x)\, dx = \delta_{mn}$$

Kronecker delta
=1 if m equal to n
=0 if m diff from n

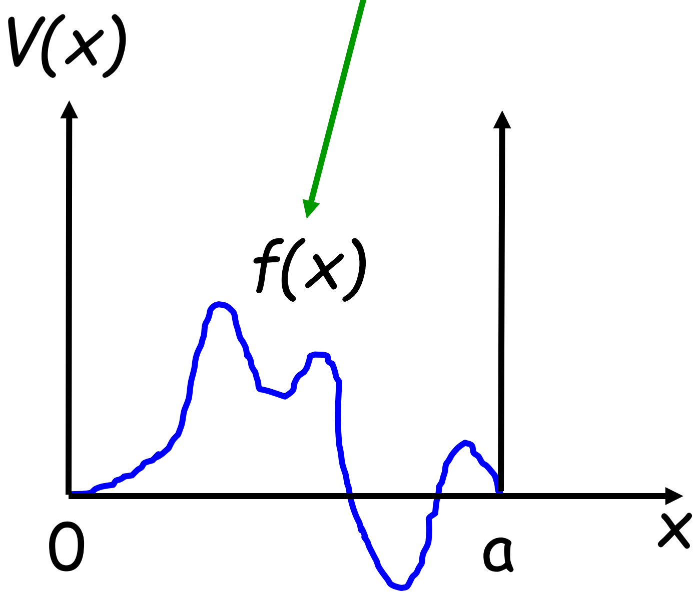
If m=n this is obvious from normalization done.

If m diff n then

$$\begin{aligned}
\int \psi_m(x)^* \psi_n(x)\, dx &= \frac{2}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right) dx \\
&= \frac{1}{a}\int_0^a \left[\cos\left(\frac{m-n}{a}\pi x\right) - \cos\left(\frac{m+n}{a}\pi x\right)\right] dx \\
&= \left\{\frac{1}{(m-n)\pi}\sin\left(\frac{m-n}{a}\pi x\right) - \frac{1}{(m+n)\pi}\sin\left(\frac{m+n}{a}\pi x\right)\right\}\Bigg|_0^a \\
&= \frac{1}{\pi}\left\{\frac{\sin[(m-n)\pi]}{(m-n)} - \frac{\sin[(m+n)\pi]}{(m+n)}\right\} = 0.
\end{aligned}$$

(II) They are complete. This means coefficients c_n can always be found such that any wave function inside the square well can be written as

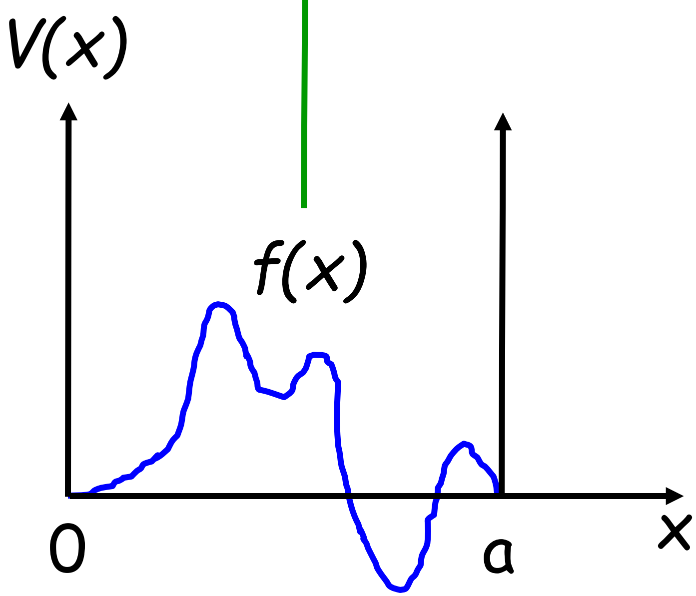
$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) = \underbrace{\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} x\right)}_{\text{Here this property is not surprising. This is just the Fourier series of f(x).}}$$



f(x) can be **ANY function that is 0 outside the well**. If not, then it is not acceptable. It can also be discontinuous inside the well.

How do we find the coefficients?

$$\int \psi_m(x)^* f(x)\,dx = \sum_{n=1}^{\infty} c_n \int \psi_m(x)^* \psi_n(x)\,dx = \sum_{n=1}^{\infty} c_n \delta_{mn} = c_m$$

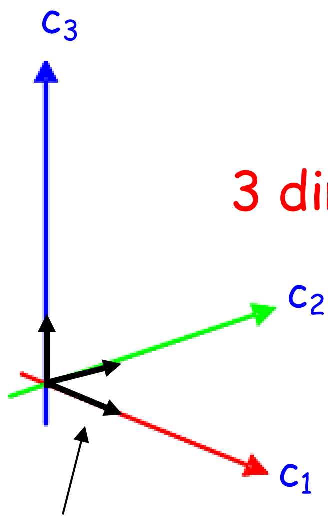


$$\boxed{c_n = \int \psi_n(x)^* f(x)\,dx.}$$

The integration can be done analytically or numerically.

Cartesians axes

$$\mathbf{r} = \sum_{n=1}^{3} c_n\, \mathbf{e}_n$$



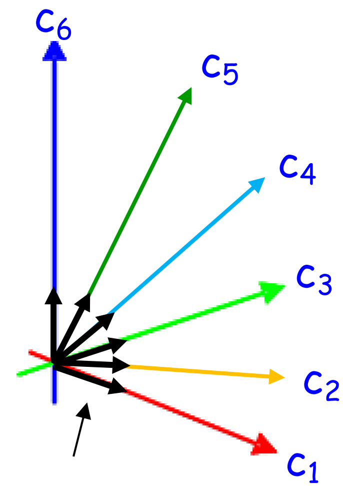
3 dimensions

Unit vectors e_1,e_2,e_3.

Any vector can be expanded in the orthonormal basis e_1,e_2,e_3.

Square well solutions

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$



∞ dimensions

"Unit vectors" are $\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\ldots$

Any wave function can be expanded in the orthonormal basis ψ_n

All these properties are not pathological of the square well but very generic.

(3) Continuation from page 27 Ch2. There are several possible values of E, say E_1, E_2, E_3, …, as found in example. For each "allowed" energy, there is a solution of time-indep. Sch. Eq.

$$\Psi_1(x,t) = \psi_1(x)e^{-iE_1t/\hbar}, \quad \Psi_2(x,t) = \psi_2(x)e^{-iE_2t/\hbar}, \ \ldots .$$

Make a linear combination:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_nt/\hbar}$$

Statement: any wave function $\Psi(x,t)$ can be written as above. The c_n's are the same as before i.e. time INDEPENDENT. But the linear combination above is NOT a stationary state.

Not in book:

Is this a solution of the time-dep. Sch. Eq. with V(x)?

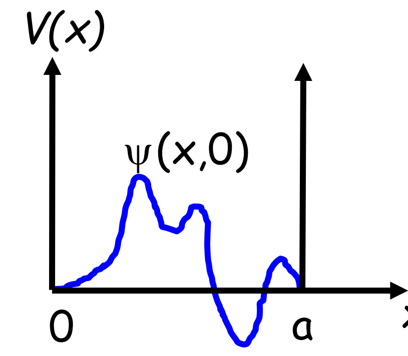
$$\hat{H}\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\hat{H}\psi_n(x)}_{E_n\psi_n(x)} e^{-iE_nt/\hbar} \qquad \text{(a)}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x) \underbrace{i\hbar\frac{\partial}{\partial t}e^{-iE_nt/\hbar}}_{E_n e^{-iE_nt/\hbar}} \qquad \text{(b)}$$

(a) = (b) → The linear combination is solution of the time-dependent Sch. Eq.

Final recipe for $\Psi(x,t)$ in square well:

Given an arbitrary $\Psi(x,0)$ -- that satisfies the BC -- you want $\Psi(x,t)$.



(1) Find stationary states and energies.

(2) "Somehow" do the integrals for coefficients:

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

Done!

$$\boxed{\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \overbrace{e^{-i(n^2\pi^2\hbar/2ma^2)t}}^{e^{-iE_n t/\hbar}}}}$$

The procedure is general but of course $\psi_n(x)$ and E_n are diff for diff potentials

Example 2.1. Assume you are given at t=0:

Real for simplicity.

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$$

Stationary states sin(nπx/a). NOT any arbitrary function like, say, $e^{-|x|}$

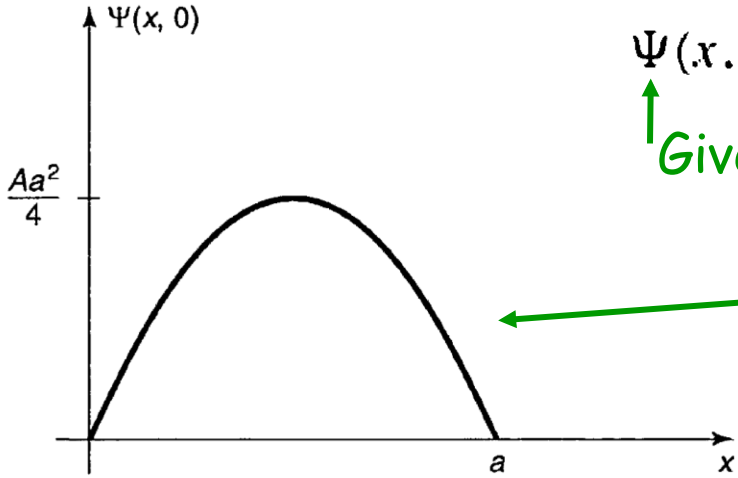
$$\Psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

$$|\Psi(x,t)|^2 = (c_1\psi_1 e^{iE_1t/\hbar} + c_2\psi_2 e^{iE_2t/\hbar})(c_1\psi_1 e^{-iE_1t/\hbar} + c_2\psi_2 e^{-iE_2t/\hbar})$$

$$= c_1^2\psi_1^2 + c_2^2\psi_2^2 + 2c_1c_2\psi_1\psi_2\cos[(E_2 - E_1)t/\hbar].$$

The prob. density is now time dependent even if stationary states are combined (quantum beat).

Example on how to apply the recipe:



$$\Psi(x,0) = Ax(a-x), \quad (0 \le x \le a)$$

Given

Usual A to normalize.

Even with respect to middle reflection. Naively looks like ground state but it's not.

Normalize first:

$$1 = \int_0^a |\Psi(x,0)|^2\,dx = |A|^2 \int_0^a x^2(a-x)^2\,dx \longrightarrow A = \sqrt{\frac{30}{a^5}}$$

Find coefficients c_n:

n odd only nonzero

$$c_n = \underbrace{\sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right)}_{\psi_n(x)} \underbrace{\sqrt{\frac{30}{a^5}}\, x(a-x)}_{\Psi(x,0)}\, dx \;=\; 8\sqrt{15}/(n\pi)^3$$

See integration process in book. Even n (i.e. odd functions) gives 0 by symmetry.

Then, you can write the final answer:

$$\boxed{\Psi(x,t) = \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-in^2\pi^2\hbar t/2ma^2}}$$