Having an exact solution is ideal to study the adiabatic approximation!

It is important to identify the external and internal characteristic times:

$$T_e = 1/\omega$$

$$\mathbf{B}(t) = B_0[\sin\alpha\cos(\omega t)\hat{\imath} + \sin\alpha\sin(\omega t)\hat{\jmath} + \cos\alpha\hat{k}]$$

$$T_i = \hbar/(E_+ - E_-) = 1/\omega_1$$

$$E_\pm = \pm\frac{\hbar\omega_1}{2} \qquad \omega_1 \equiv \frac{eB_0}{m}$$

The adiabatic approximation is when $T_e \gg T_i$, namely $\omega \ll \omega_1$.

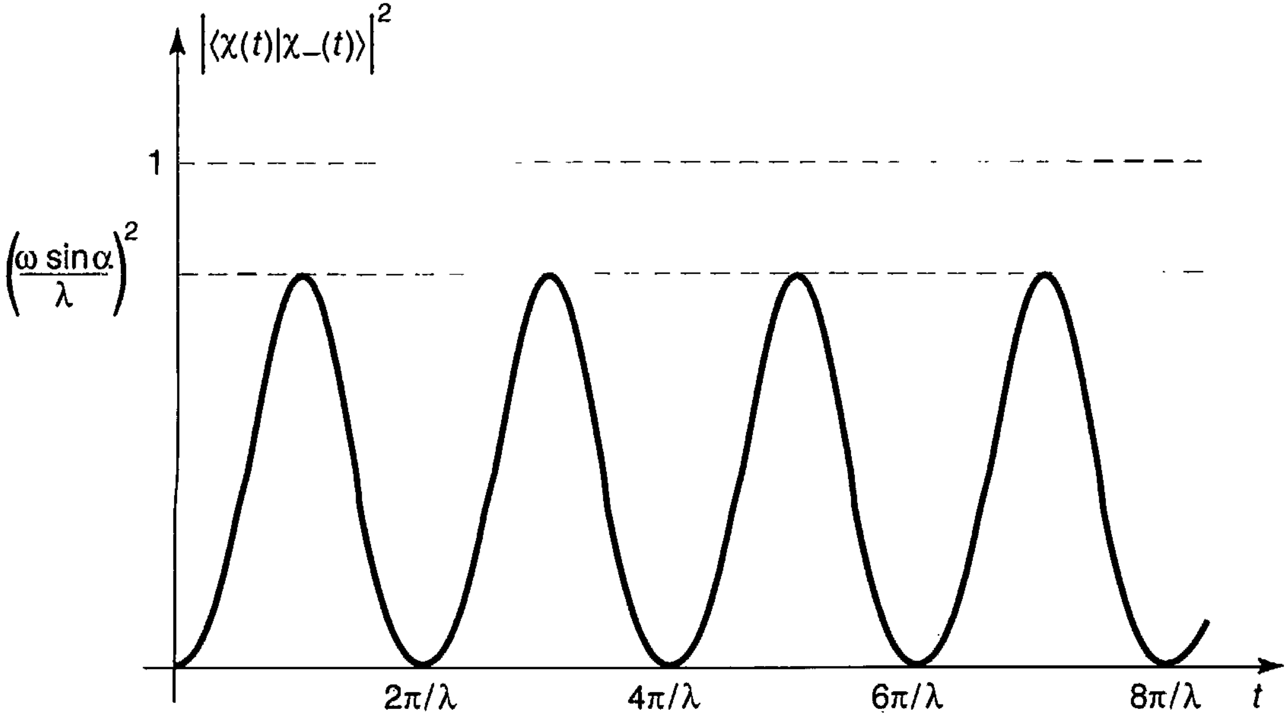
$$|\langle\chi(t)|\chi_-(t)\rangle|^2 = \left[\frac{\omega}{\lambda}\sin\alpha\sin\left(\frac{\lambda t}{2}\right)\right]^2 \cong \left[\frac{\omega}{\omega_1}\sin\alpha\sin\left(\frac{\lambda t}{2}\right)\right]^2 \rightarrow 0$$

If $\frac{\omega}{\omega_1} \rightarrow 0$ then $\lambda \equiv \sqrt{\omega^2 + \omega_1^2 - 2\omega\omega_1\cos\alpha} \longrightarrow \omega_1$

In the adiabatic limit the magnetic field leads the electron "by its nose" to rotate its orientation all the time pointing along B(t).

For completeness, see Example 10.2 to separate dynamic vs geometric phases.

$$|\langle \chi(t)|\chi_{-}(t)\rangle|^2 = \left[\frac{\omega}{\lambda}\sin\alpha\sin\left(\frac{\lambda t}{2}\right)\right]^2$$



In general the probability of "down spin" will not be zero. It will oscillate. But weight is regulated by ω/λ. Thus, in adiabatic limit the amplitude will be minuscule.

Chapter 8: The WKB Approximation

Back to time independent problems. WKB stands for Wentzel, Kramers, Brillouin.

WKB is a technique to obtain approximate solutions to time independent problems, mainly in 1D or where only "r" matters in 3D.

Main intuitive idea: suppose you have a potential V(x) totally constant, no imperfections. Then, the solution if $E > V(x)$ is:

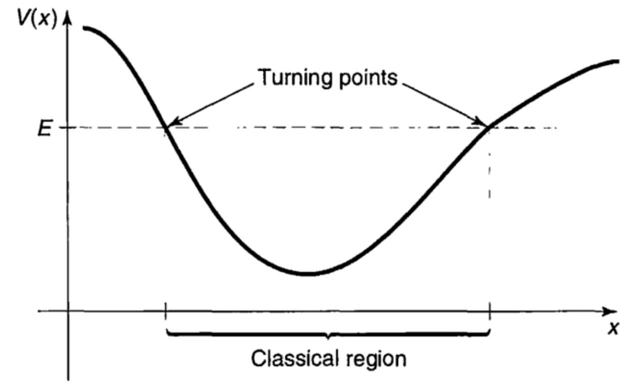
$$\psi(x) = Ae^{\pm ikx} \qquad k \equiv \sqrt{2m(E-V)}/\hbar \qquad \lambda = 2\pi/k$$

Of course, here *A* is constant, *k* is constant, λ is constant.

However, a perfect flat potential is unlikely. Suppose V(x) is "nearly" flat but changes very slowly with x, i.e. over distances much larger than λ. Then, the solution cannot be too different: *A*, k,… will now be smooth slowly varying functions of x.

8.1: The "Classical" Region

Let us first consider the case E>V(x), i.e. the classical region. First, we will not make any approximation and find exact equations for amplitude and phase. Then, we will make the WKB approximation.



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \qquad E > V(x)$$

Exactly, this can be written $\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi$ where $p(x) \equiv \sqrt{2m[E - V(x)]}$

Propose $\psi(x) = A(x)e^{i\phi(x)}$, which is generic for any wave function. Here both *A(x)* and ϕ *(x)* are real functions.

$$\frac{d\psi}{dx} = (A' + iA\phi')e^{i\phi}$$

$$\frac{d\psi}{dx} = (A' + iA\phi')e^{i\phi} \longrightarrow \frac{d^2\psi}{dx^2} = [A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2]e^{i\phi}$$

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi \longrightarrow A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2 = -\frac{p^2}{\hbar^2}A$$

$$p(x) \equiv \sqrt{2m[E - V(x)]}$$

Real part:

$$A'' - A(\phi')^2 = -\frac{p^2}{\hbar^2}A$$

$$A'' = A\left[(\phi')^2 - \frac{p^2}{\hbar^2}\right]$$

Cannot be solved unless we assume A''~0, i.e. amplitude varies slowly with x.

Imaginary part:

$$2A'\phi' + A\phi'' = 0$$

$$\left(A^2\phi'\right)' = 0$$

$$A^2\phi' = C^2$$

Exact: $$A = \frac{C}{\sqrt{\phi'}}$$

Again, the two exact eqs. are:

$$A'' = A\left[(\phi')^2 - \frac{p^2}{\hbar^2}\right] \qquad A = \frac{C}{\sqrt{\phi'}}$$

If A″=0, then:

$$(\phi')^2 = \frac{p^2}{\hbar^2}$$

$$\phi(x) = \pm\frac{1}{\hbar}\int p(x)\,dx$$

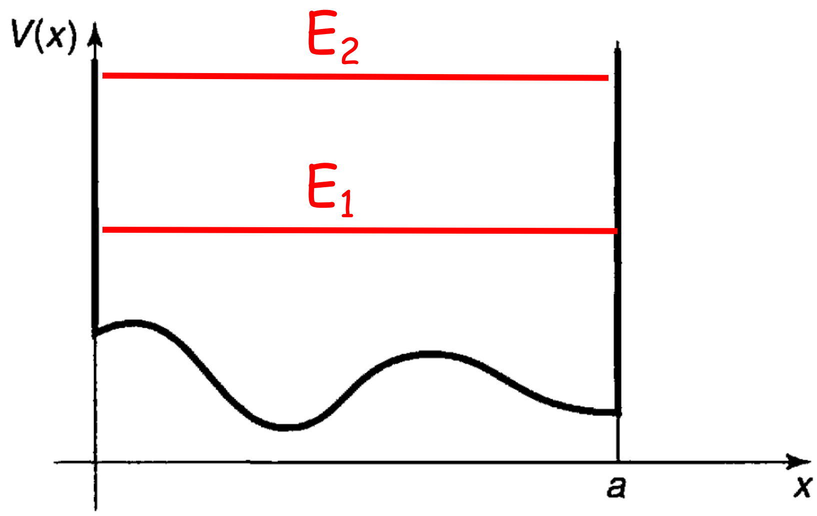
We started with $\psi(x) = A(x)e^{i\phi(x)}$ then we arrive to:

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int p(x)\,dx}$$

This is the WKB approximation to the wave function.

Note $\phi(x)$ is an indefinite integral i.e. x dependent.
We will need boundary conditions.

Example 8.1: Potential well with two vertical walls.



Assume E > V(x) for all values of x
(this may or may not be right, we have to be careful).

We found before:

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)\,dx}$$

In general, we have to make a linear combination:

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}} \left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)} \right]$$

where $\phi(x) = \frac{1}{\hbar} \int_0^x p(x')\,dx'$

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)}\right]$$

↓

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}[C_1 \sin\phi(x) + C_2 \cos\phi(x)]$$

Boundary conditions:

(1) $\psi(x) = 0$ at $x = 0$

This means $C_2 = 0$

(2) $\psi(x) = 0$ at $x = a$

This means $\phi(a) = n\pi$ $(n = 1, 2, 3, \ldots)$

$$\phi(a) = n\pi \quad (n = 1, 2, 3, \ldots)$$

means

$$\int_0^a p(x)\,dx = n\pi\hbar$$

$$\int_0^a \sqrt{2m[E - V(x)]}\,dx = n\pi\hbar$$

where E is the unknown for each "n".

The integral can be done analytically and an equation for E will be found (see HW23, 8.1), or we can find E numerically.

If V(x)=0 inside the well, then of course:

$$\int_0^a \sqrt{2m[E - V(x)]} = \int_0^a \sqrt{2m\ E}\ dx = n\pi\hbar$$

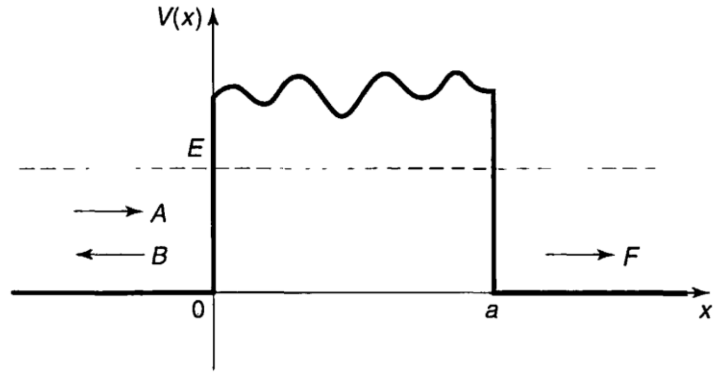
$$\sqrt{2m\ E}\ a = n\pi\hbar$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

which is the exact result.

8.2: "Tunneling"

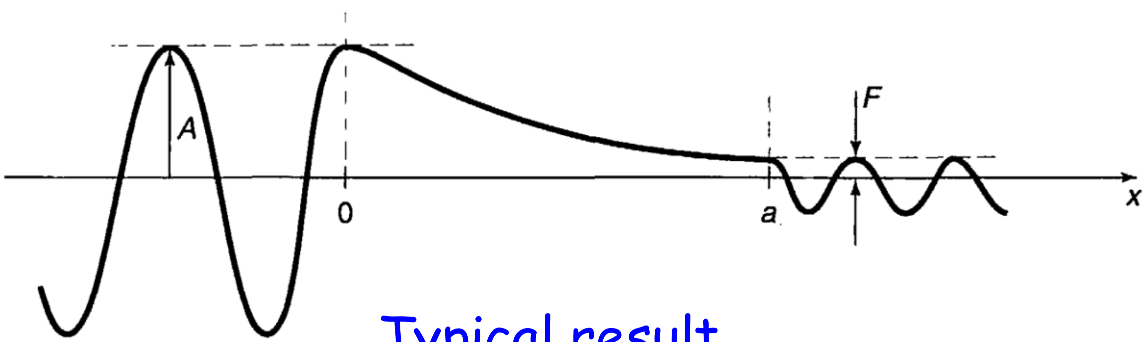
Now consider regions that are NOT classical i.e. E<V(x).



We can repeat all the same and we find:

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm \frac{1}{\hbar} \int |p(x)|\, dx}$$

Note: no "*i*" in phase and |..| in *p(x)*



Typical result

$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar} \int_0^a |p(x')|\, dx'}$$