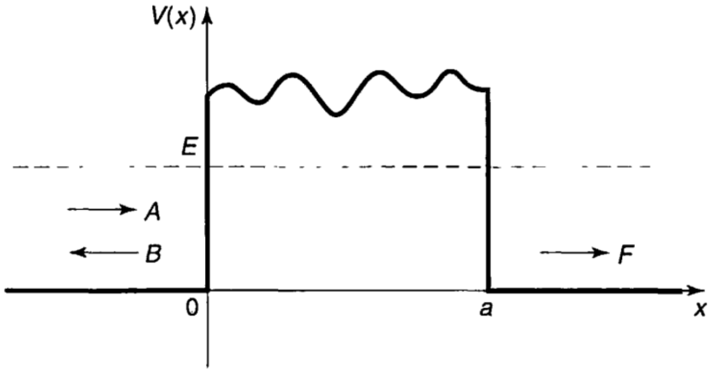
8.2: "Tunneling"

Now consider regions that are NOT classical i.e. E<V(x).



Indefinite integral, i.e. x dependent

We can repeat all the same and we find:

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm\frac{1}{\hbar}\int |p(x)|\,dx}$$

Note: no "*i*" in phase and |..| in *p*(*x*)

Typical result

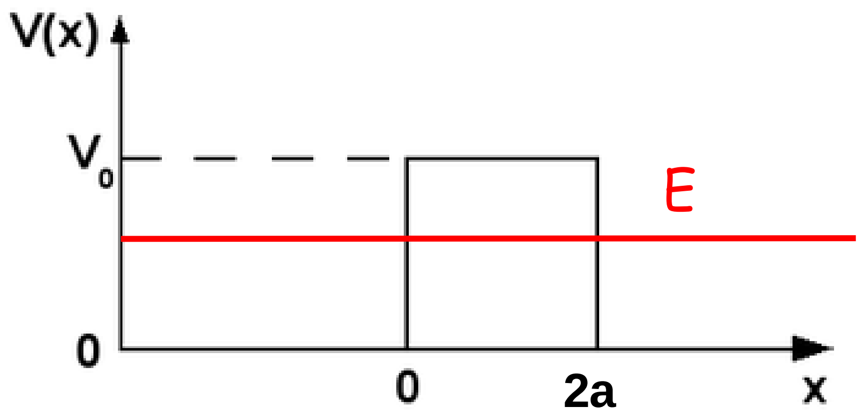
$$\frac{|F|}{|A|} \sim e^{-\frac{1}{\hbar}\int_0^a |p(x')|\,dx'}$$

The result of previous page is a general result:

$$\frac{|F|^2}{|A|^2} \sim e^{-\frac{1}{\hbar}2\int_0^a |p(x')|\,dx'}$$

$$\boxed{T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx}$$

Example,
Problem 8.3:



This is the **exact result** from Ch. 2:

$$T_{\text{exact}} = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\gamma}$$

where

$$\gamma = \frac{2a}{\hbar}\sqrt{2m(V_0 - E)}$$

General WKB approx. for tunneling through barrier of width a:

We pretend we do not know the exact result and try to use the WKB approximation:

$$\boxed{T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx}$$

Width of barrier is $2a$ here:

$$\gamma = \frac{1}{\hbar}\int |p(x)|\,dx = \frac{1}{\hbar}\int_0^{2a} \sqrt{2m(V_0-E)}\,dx = \frac{2a}{\hbar}\sqrt{2m(V_0-E)}$$

Then, WKB prediction for tunneling is:

$$\boxed{T \approx e^{-4a\sqrt{2m(V_0-E)}/\hbar}}$$

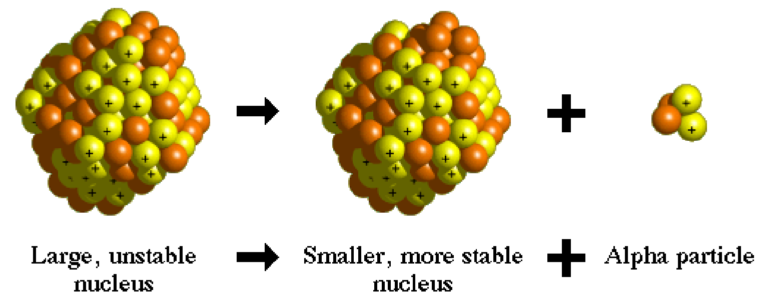
$$\underset{\text{exact}}{T} = \frac{1}{1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\gamma} \quad \Bigg| \quad \sinh\gamma = \tfrac{1}{2}(e^{\gamma}-e^{-\gamma}) \approx \tfrac{1}{2}e^{\gamma} \quad \Bigg| \quad \sinh^2\gamma \approx \tfrac{1}{4}e^{2\gamma},$$

$$\underset{\text{exact}}{T} \approx \frac{1}{1+\frac{V_0^2}{16E(V_0-E)}e^{2\gamma}} \approx \boxed{\left\{\frac{16E(V_0-E)}{V_0^2}\right\} e^{-2\gamma}}$$

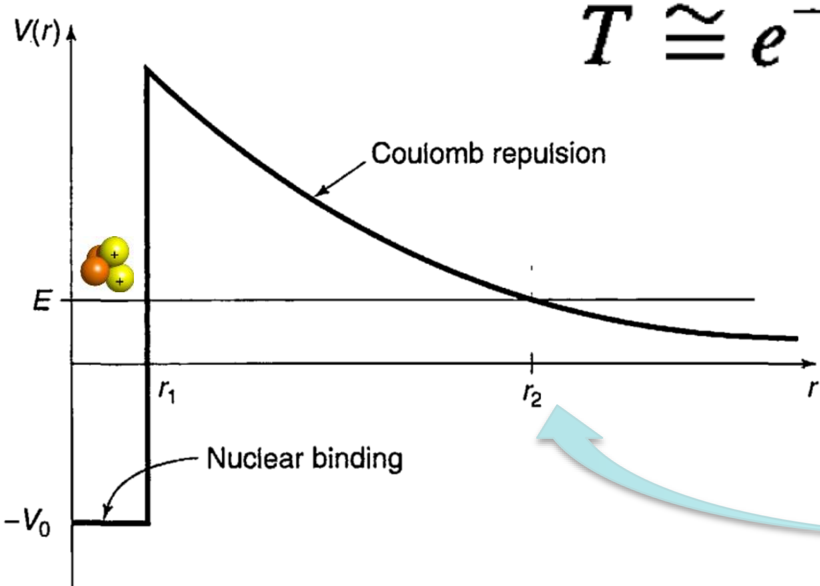
$$\underset{\text{exact}}{\gamma} = \frac{2a}{\hbar}\sqrt{2m(V_0-E)}$$

Dominant exponential is properly reproduced by WKB

Famous example: Gamow's theory of alpha decay (1928)



$$T \cong e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$



$$\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r_2} = E$$

size $r_1 \sim 10^{-15}$ m

$$\boxed{\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2}\sqrt{2m\left(\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - E\right)}\,dr}$$

Turns out, the integral can be done exactly, and moreover it can be simplified considerably if $r_1 \ll r_2$

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2} \sqrt{2m\left(\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - E\right)}\,dr \longrightarrow K_1 \frac{Z}{\sqrt{E}} - K_2\sqrt{Zr_1}$$

Full integral

After some approximations

where

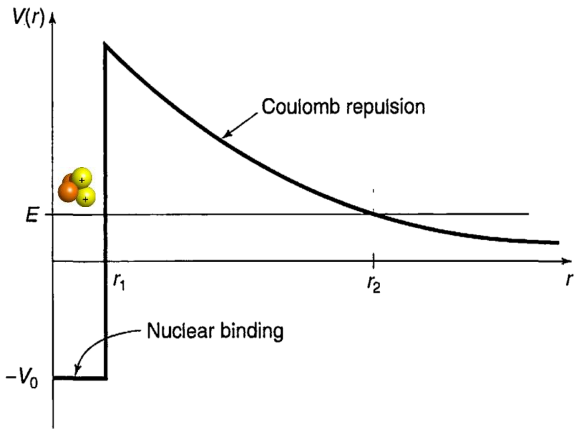
$$K_1 \equiv \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\pi\sqrt{2m}}{\hbar} = 1.980\ \mathrm{MeV}^{1/2}$$

$$K_2 \equiv \left(\frac{e^2}{4\pi\epsilon_0}\right)^{1/2}\frac{4\sqrt{m}}{\hbar} = 1.485\ \mathrm{fm}^{-1/2}$$

1 fm = 10^{-15} m is the size of a typical nucleus

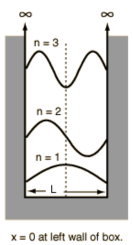
As "m" we use the mass of an alpha particle ~ 4 proton masses

Z is the positive charge of the nucleus



If alpha particles have an average velocity "v" inside the well, then to travel from r=0 to r=r_1 it takes t=r_1/v, i.e. hits the walls with a period 2r_1/v. At each collision the probability of **remaining trapped** is $e^{+2\gamma}$ (or prob. of escape is $e^{-2\gamma}$)

The **lifetime** then is $\tau = (2r_1/v)\, e^{+2\gamma}$.

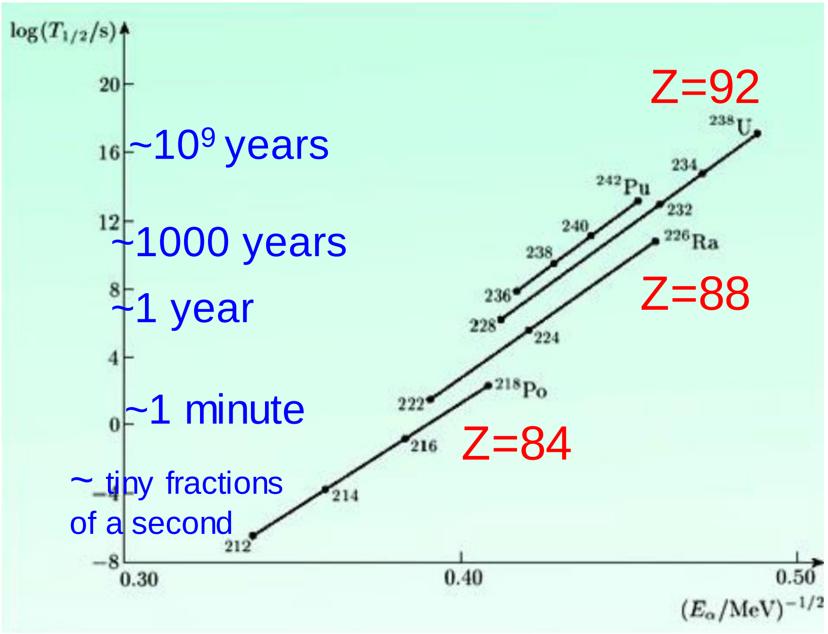


The energy of the alpha particle is not arbitrary but resembles that of a square well.

Then, $\ln(\tau) = \ln(2r_1/v) + 2\gamma$ with

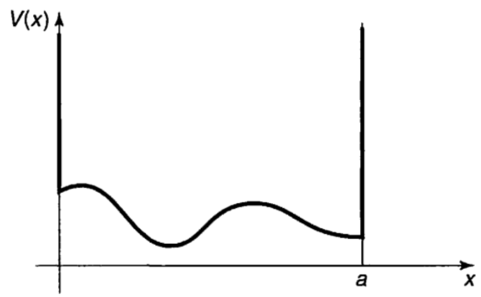
$$\gamma \sim K_1 \frac{Z}{\sqrt{E}} - K_2\sqrt{Zr_1}$$

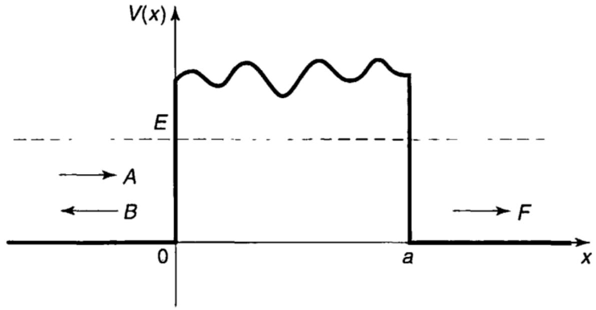
Experiments confirm that lifetime depends ~ linearly on 1/sqrt{Eα} on a range of lifetimes from 10^9 years to tiny fractions of seconds!
(Geiger-Nuttall law)



8.3: The connection region

In many examples we use the WKB approximation in cases V(x) has vertical walls.

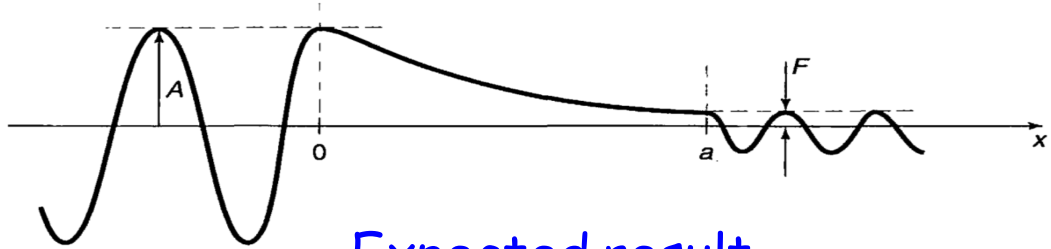




But in most real situations, this is not the case, such as in alpha decay. We may try the "usual" procedure:

$$\psi(x) \cong \begin{cases} \dfrac{1}{\sqrt{p(x)}}\left[Be^{\frac{i}{\hbar}\int_x^0 p(x')\,dx'} + Ce^{-\frac{i}{\hbar}\int_x^0 p(x')\,dx'}\right], & \text{if } x < 0, \\ \dfrac{1}{\sqrt{|p(x)|}} De^{-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'}, & \text{if } x > 0. \end{cases}$$

Naively we may simply be tempted to try to match coefficients at the boundary.



Expected result

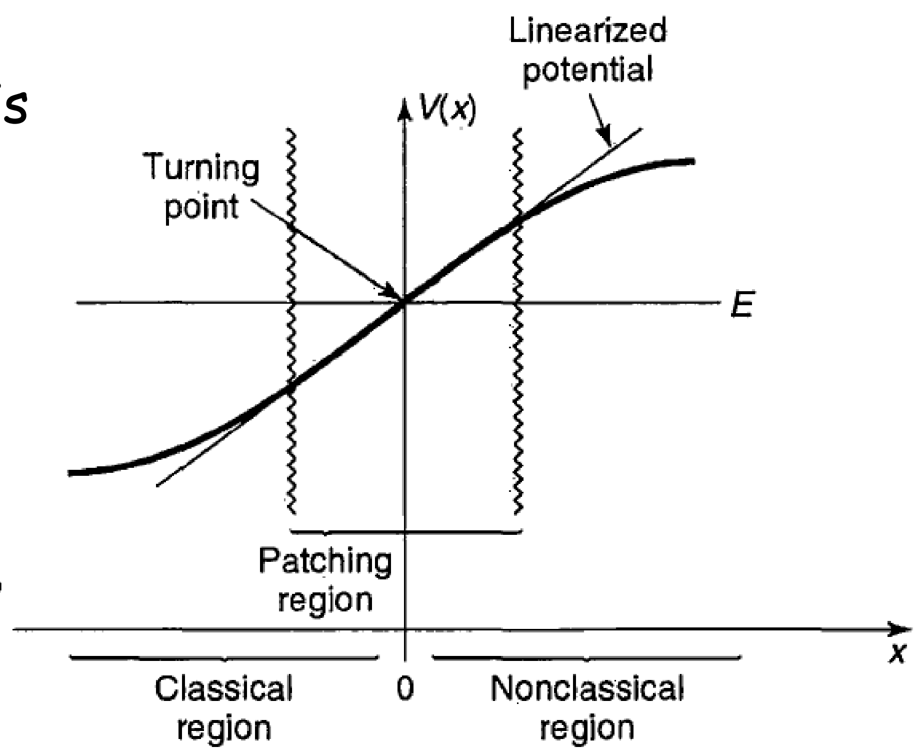
However, at exactly the "x" where we switch from classical to non-classical then p(x) = V(x)-E is zero. Then, WKB wave functions explode. Not realistic!

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm\frac{1}{\hbar}\int |p(x)|\,dx}$$

In practice a "patching procedure" is followed, where a "third region" is introduced where the potential is linearized

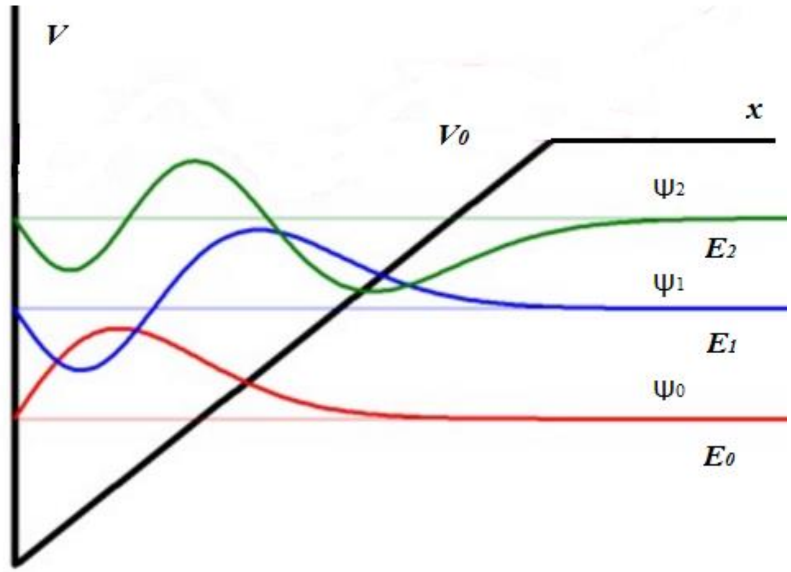
$$V(x) \cong E + V'(0)x$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_p}{dx^2} + [E + V'(0)x]\psi_p = E\psi_p$$



A problem with a linear potential is exactly solvable and leads to the Airy functions, complicated functions usually given in an integral form. They are oscillatory on one side and exponential on the other.

If we had a sharp wall on one side (not the actual problem at hand) the shape of the Airy functions is as shown (leading to bound states):



The WKB patching procedure would be too complicated to describe in detail, just be aware of its existence.