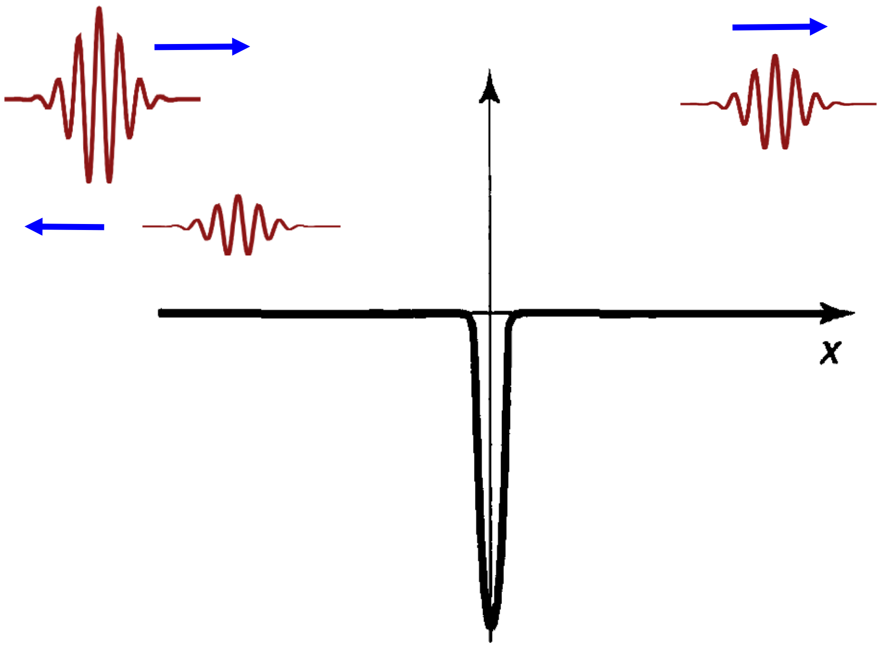
4.4: Spin of electrons (here we restart QM 412)

This topic is not part of Test 3 QM-411, but it will be part of Test 1 of QM-412

A classical rigid body, like a planet, can have two kinds of angular momenta: (1) **L**, the orbital one associated with the center of mass, like Earth around the sun, and (2) **S**, the spin, like Earth rotating daily about an axis.

In quantum mechanics we already discussed the orbital component **L** (related with the electron around the nucleus).

In QM, we also have a spin **S** for the electron but … the electron to the best of our accuracy is a POINT, thus cannot rotate.



However, once we measure the location and find the particle at position x_0, then that "sigma" width is gone. The particle is perfectly at x_0. At that moment what radius it has?

In wave packets as shown, the "finite size" due to "sigma" is the finite size of the wave function related to the probability of finding the particle.

It seems that the radius is smaller than 10^-18 m according to experiments (see email I sent to all). Radius of nucleus is 10^-15 m.

NEW: The spin appears naturally from the Dirac equation that links quantum mechanics and relativity.

From Wikipedia on Dirac equation:

(The Dirac eq.) provided a theoretical justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin (note: this is what we are doing); the wave functions in the Dirac theory are vectors of four complex numbers, two of which resemble the Pauli wavefunction in the non-relativistic limit, in contrast to the Schrödinger equation which described wave functions of only one complex value.

It is a fact of Nature, that elementary particles such as an electron carry an intrinsic spin angular momentum **S**.

Because the electron is a point, we cannot use the classical formulas $\mathbf{S} = I\boldsymbol{\omega}$ or $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

To describe the intrinsic spin the math has to be "analogous" to that of **L**. Let us start with the commutators:

$$\boxed{[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y}$$

becomes ...

$$\boxed{[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y}$$

The eigenfunctions are more "abstract" ...

First, let us switch to the Ch. 3 notation using an abstract Hilbert space notation:

$$\boxed{L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m}$$ to

$L^2 \,| l\ m_l > = \hbar^2\, l(l+1)\, | l\ m_l > ; \quad L_z\, | l\ m_l > = \hbar\ m_l\, | l\ m_l >$

For L^2 and L_z using $Y_l^m(\theta, \phi)$ or $| l\ m_l >$ is the SAME.

But for the intrinsic spin the Ch. 3 notation is the ONLY way because there are no angles to use.

Because in lecture Nov. 20 2018 we arrived all the way to the eigenvalues by only using the commutators, then we simply repeat the operation line by line and find:

$$\boxed{S^2 \,|s\, m_s> = \hbar^2 s(s+1)\,|s\, m_s> ; \quad S_z\,|s\, m_s> = \hbar\, m_s\,|s\, m_s>}$$

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots ; \quad m = -s, -s+1, \ldots, s-1, s$$

The spin of each type of particle is FIXED, not like the orbital angular momentum that you can change by emission or absorption of energy.

This is where the more practical portion of spin starts … our focus will be on the simplest case of $s=1/2$

4.4.1 : Spin ½ (electrons, quarks)

Use $S^2\,|s\ m_s> = \hbar^2\, s(s+1)\,|s\ m_s>$; $S_z\,|s\ m_s> = \hbar\, m_s\,|s\ m_s>$

Specialize for s=1/2. Then, there are only two states, which in abstract form (no angles θ and ϕ!) are:

$|\ \tfrac{1}{2}\ \tfrac{1}{2}>$ and $|\ \tfrac{1}{2}\ -\tfrac{1}{2}>$

We call them spin "up" or ↑ and spin "down" or ↓ .

There is another, still abstract, way to represent spins up and down. It is using so-called "**spinors**"

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is an internal degree of freedom. The two components are not a 2D vector.

We can combine the "up" and "down" linearly at will. So the spin could point "sideways" for instance.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we use spinors for the states, then what do we use for the operators such as L^2 ? Certainly we cannot use derivatives of angles. There are no angles!

From the two equations …

$$S^2 \,|\, \tfrac{1}{2}\ \ \tfrac{1}{2} > = \hbar^2\, \tfrac{1}{2}\, (\tfrac{1}{2} + 1) \,|\, \tfrac{1}{2}\ \ \tfrac{1}{2} >$$

$$S^2 \,|\, \tfrac{1}{2}\ {-\tfrac{1}{2}} > = \hbar^2\, \tfrac{1}{2}\, (\tfrac{1}{2} + 1) \,|\, \tfrac{1}{2}\ {-\tfrac{1}{2}} >$$

… it can be deduced (just apply the proposed matrix to the spinors) that:

$$\boxed{\mathbf{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

From the other two equations …

$$S_z \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2}\rangle = \tfrac{1}{2}\hbar \,|\, \tfrac{1}{2} \;\; \tfrac{1}{2}\rangle$$

$$S_z \,|\, \tfrac{1}{2} \; -\tfrac{1}{2}\rangle = -\tfrac{1}{2}\hbar \,|\, \tfrac{1}{2} \; -\tfrac{1}{2}\rangle$$

… it can deduced (again, apply the proposed matrix to the spinors) that:

$$\boxed{\mathbf{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

There has to be also an analog of the raising and lowering operators:

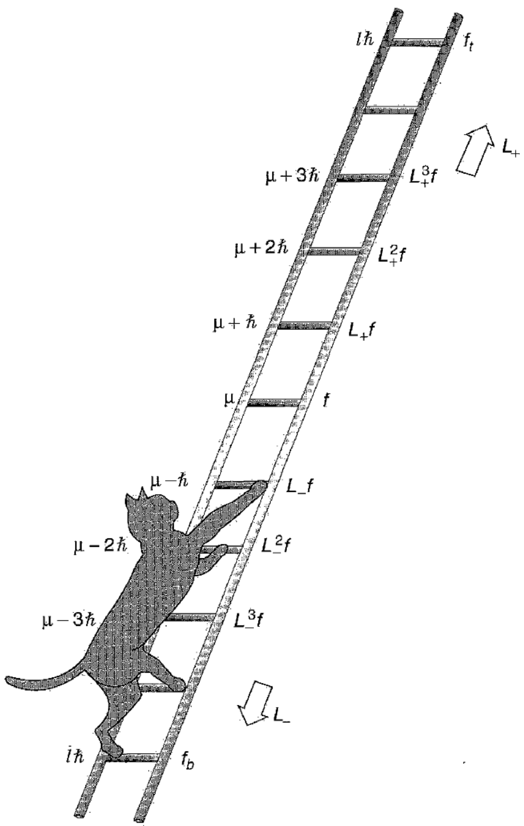
$$\mathbf{S}_+\chi_- = \hbar\chi_+$$

$$\mathbf{S}_-\chi_+ = \hbar\chi_-$$

$$\mathbf{S}_+\chi_+ = \mathbf{S}_-\chi_- = 0$$

$$\boxed{L_\pm \equiv L_x \pm iL_y}$$

$$\boxed{\begin{aligned}\mathbf{S}_+ &= \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\ \mathbf{S}_- &= \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\end{aligned}}$$



Example:

$$\hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \hbar\begin{pmatrix}1\\ 0\end{pmatrix}$$

Recalling $S_\pm = S_x \pm i S_y$ then:

$$S_x = (1/2)(S_+ + S_-) \qquad S_y = (1/2i)(S_+ - S_-)$$

$$\boxed{\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Dropping the ħ/2 factor defines
the famous **Pauli matrices**:

$$\boxed{\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Returning to the general combination:

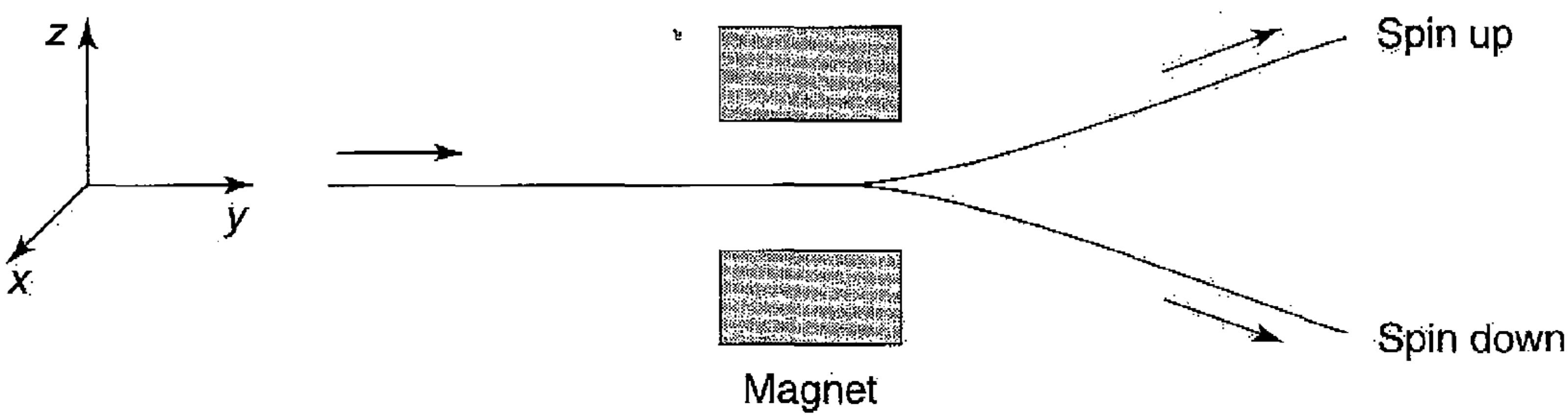
$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

You have to normalize i.e. $|a|^2 + |b|^2 = 1$

$|a|^2$ is the probability of measuring spin up.

$|b|^2$ is the probability of measuring spin down.

How do you measure a spin? The same as any magnetic moment, like the orbital "l". You introduce the particle in a magnetic field. Also there is something called the Stern-Gerlach experiment (to be explained next week):



Returning, again!, to the general combination:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

What is the probability that the spin points say along the *positive x axis*?

To answer this question, first you have to diagonalize the 2x2 Pauli matrix "x" and find the "eigenspinors".

For x Pauli matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$+\frac{\hbar}{2} \quad \text{Eigenvalues} \quad -\frac{\hbar}{2}$$

For z Pauli matrix:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+\frac{\hbar}{2} \quad \text{Eigenvalues} \quad -\frac{\hbar}{2}$$

Make sure you know how to find eigenspinors!

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

x Pauli matrix

construct determinant

solve determinant; find eigenvalues

Finally find eigenvectors:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\beta = \pm\alpha$$

... and finally normalize to 1.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

In HW13 you have to repeat for the y Pauli matrix

The general combination …

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

… can now be written as:

$$\boxed{\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}}$$

$$(1/2)|a+b|^2$$

is the probability of measuring spin up along x.

$$(1/2)|a-b|^2$$

is the probability of measuring spin down along x.

QM 412 true new material starts here. Page 175 book, second edition

Let us see how the formulas work in practice. Start with:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider example 4.2 book. The arbitrary spin state given is

$$\boxed{\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}}$$

Then, $a = (1+i)/\sqrt{6}$ and $b = 2/\sqrt{6}$

prob. of $+\hbar/2$ if S_z measured $= |(1+i)/\sqrt{6}|^2 = 1/3$

prob. of $-\hbar/2$ if S_z measured $= |2/\sqrt{6}|^2 = 2/3$

Consider now the SAME spinor but from the perspective of the eigenspinors of the x Pauli matrix. You can use any basis after all.

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$+\hbar/2$ $\qquad$ $-\hbar/2$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$a = (1+i)/\sqrt{6}$$
$$b = 2/\sqrt{6}$$

$$\left|\frac{a+b}{\sqrt{2}}\right|^2 = \text{prob. of getting } +\hbar/2 \text{ if } S_x \text{ is measured} = (1/2)|(3+i)/\sqrt{6}|^2 = 5/6$$