$$\left|\frac{a-b}{\sqrt{2}}\right|^2$$ = prob. of getting $-\hbar/2$ if S_x is measured = $(1/2)|(-1+i)/\sqrt{6}|^2 = 1/6$

prob. of getting $+\hbar/2$ or $-\hbar/2$ if S_x is measured has to be 1.

Indeed 5/6 + 1/6 =1.

What is the "expectation value" of S_x ?

$$\frac{5}{6}\left(+\frac{\hbar}{2}\right)+\frac{1}{6}\left(-\frac{\hbar}{2}\right)=\frac{\hbar}{3}$$

Alternatively, you get the same ħ/3 as follows:

Check last step! $(e,f)\begin{pmatrix} c \\ d \end{pmatrix} = e.c+f.d$

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

σ_x Pauli matrix

$$\frac{\hbar}{2}\begin{pmatrix} 2/\sqrt{6} \\ (1+i)/\sqrt{6} \end{pmatrix}$$

$$\langle S_x \rangle = \chi^\dagger \mathbf{S}_x \chi = \left(\frac{(1-i)}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}\begin{pmatrix} (1+i)/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} = \frac{\hbar}{3}$$

horizontal spinor with each component conjugated

vertical spinor as given

4.4.2 Electron in a magnetic field

When an electron is at rest inside a uniform magnetic field $\mathbf{B}$, the Hamiltonian is

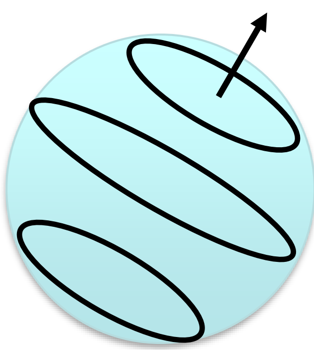
$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma \mathbf{B} \cdot \mathbf{S}$$

$\boldsymbol{\mu} = \gamma\, \mathbf{S}$

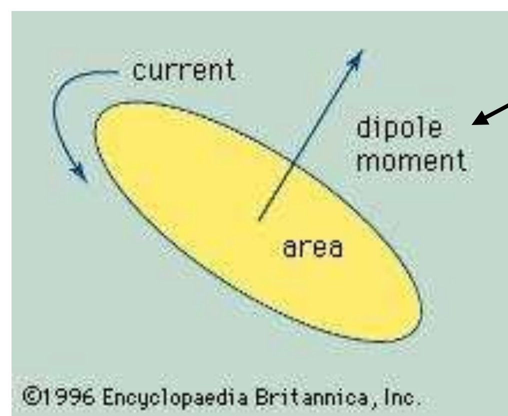
γ = "gyromagnetic ratio" =$-|e|/m$ for electrons (factor 2 difference with classical calculation due to relativity)

A spinning charged object is made of little loops of current.

Each little loop generates a little magnetic field. They add up to a net dipole magnetic moment.



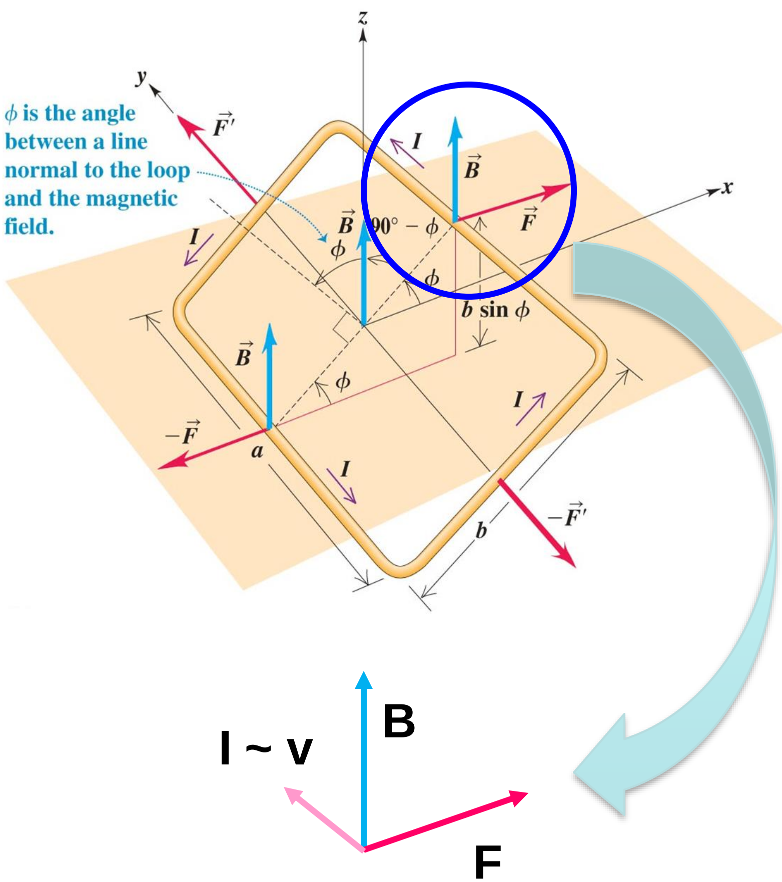
total charge q
total mass m



small μ

If q<0, then current and dipole invert; μ and S are antiparallel

When a little loop of current is introduced in a strong external magnetic field, the Lorentz force on the moving charges generates a torque.



Lorentz force
F ~ v x B

The torque tries to align the dipole moment with B, and the energy is found to be in E&M

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}$$

Suppose for simplicity that the field points along the z axis i.e. $\mathbf{B} = (0,0,1)B_0$. Then,

$$\mathbf{H} = -\gamma B_0 \mathbf{S}_z = -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Change of notation only to emphasize 2x2 character. Optional!

Eigenfunctions, or eigenspinors, are:

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{H} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(\gamma B_0 \hbar)/2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Energy eigenvalue E_+

For spin down, the energy E_- changes sign.

The Hamiltonian is time independent because the magnetic field is constant. Actually, all Hamiltonians we studied in QM411 were time independent.

If H is time independent, then the general solution is a linear combination of stationary states. In the square well potential there were infinite number of states. Remember Ch2 QM411 and Tests 1 and 3!

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

Focusing on S=1/2, all much easier. Just two states!

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar} = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

The values of *a* and *b* are fixed by initial condition at $t=0$, as we did in Ch. 2 for the coefficients c_n.

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

We also need to normalize: $|a|^2 + |b|^2 = 1$

When we have 2 unknowns linked as in the normalization condition, we can parametrize with an angle as $a = \cos(\alpha/2)$ $\quad$ $b = \sin(\alpha/2)$

because

$$|a|^2 + |b|^2 = \cos^2(\alpha/2) + \sin^2(\alpha/2) = 1$$

Review from QM411:

Is this a solution of the time-dep. Sch. Eq. with V(x)?

$$\hat{H}\Psi(x,t) = \sum_{n=1}^{\infty} c_n \underbrace{\hat{H}\psi_n(x)}_{E_n\psi_n(x)} e^{-iE_n t/\hbar} \quad \textbf{(a)}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) \underbrace{i\hbar\frac{\partial}{\partial t} e^{-iE_n t/\hbar}}_{E_n e^{-iE_n t/\hbar}} \quad \textbf{(b)}$$

(a) = (b) → The linear combination is solution of the time-dependent Sch. Eq.

Then, we arrive to a simple formula:

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

Physical meaning of angle α? Let us repeat what we did before in the example $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$, but now for the spinor in a magnetic field.

$$\langle S_x \rangle = \chi(t)^\dagger \mathbf{S}_x \chi(t) = \left(\cos(\alpha/2)e^{-i\gamma B_0 t/2},\ \sin(\alpha/2)e^{i\gamma B_0 t/2}\right)$$

$$\times \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$$

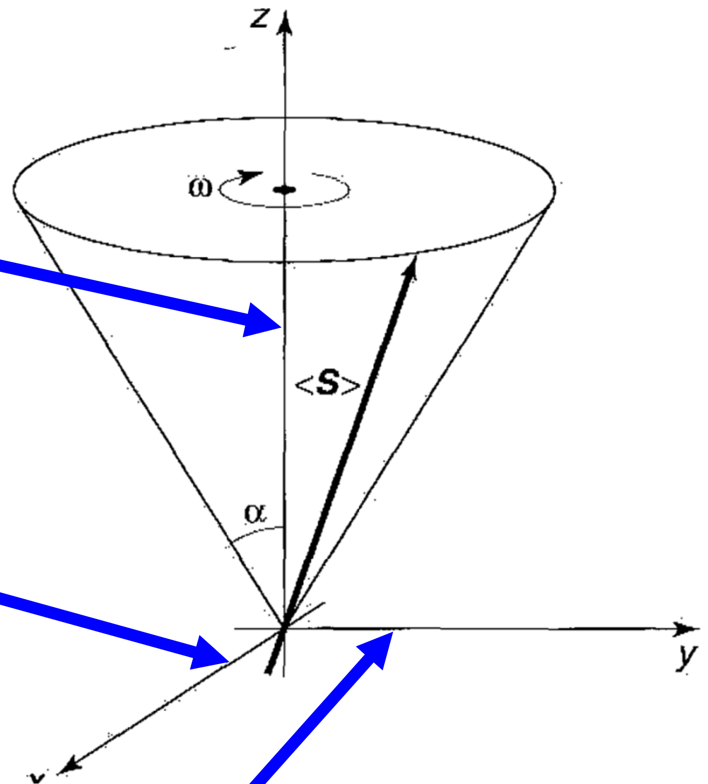
$$= \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t).$$

Make sure you confirm this calculation.

$$\langle S_z \rangle = \chi(t)^\dagger \mathbf{S}_z \chi(t) = \frac{\hbar}{2} \cos\alpha$$

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\alpha \cos(\gamma B_0 t)$$

$$\langle S_y \rangle = \chi(t)^\dagger \mathbf{S}_y \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B_0 t)$$



This is the same as a classical dipole oscillating in a magnetic field. It is precessing …

Not in book:

Why the electron precesses instead of simply aligning with the magnetic field to minimize energy?

REASON: the energy is conserved -- the electron cannot change its energy -- because the Hamiltonian is time independent.

Consider $\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix}$

$$\langle \mathbf{H} \rangle = \chi(t)^{\dagger}\, \mathbf{H}\, \chi(t) = E_+ \cos^2(\alpha/2) + E_- \sin^2(\alpha/2)$$

where $E_+ = -(\gamma B_0 \hbar)/2$
$E_- = +(\gamma B_0 \hbar)/2$

Time independent! The energy does not change in a t-independent Hamiltonian