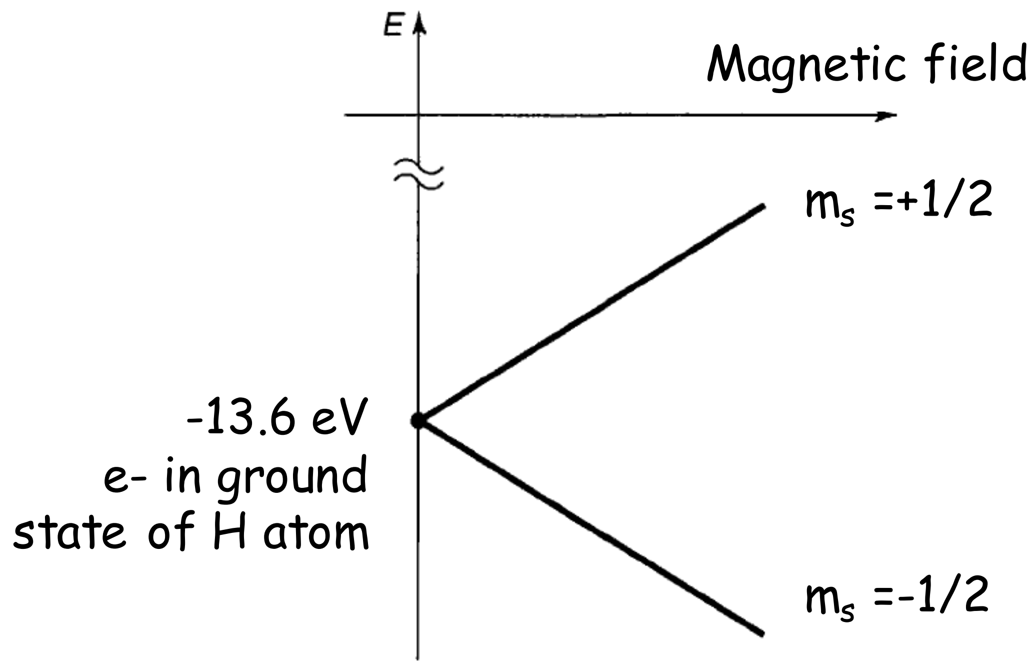
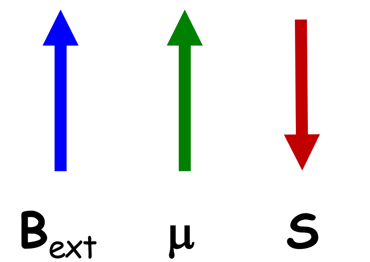
# Anticipating what is coming in Ch. 6:




$H = (|e|/m)\ \mathbf{B}_{ext} . \mathbf{S}$

# 4.4.3 Addition of angular momenta

Suppose we have TWO particles, each with spin ½.

We focus on the spin, not the actual relative motion.
It could be two electrons or one electron and one proton.

Since each spin can be up or down, then overall, regardless of actual distance, we have 4 combinations:

$$\uparrow\uparrow, \quad \uparrow\downarrow, \quad \downarrow\uparrow, \quad \downarrow\downarrow$$

We want to know: what is the total spin? Natural answers are 1=1/2 +1/2 or 0=1/2-1/2. But why not other numbers? Classically, any number between 1 and 0 is good.

means: acts only on particle 1

The total spin operator is: $\mathbf{S} \equiv \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$

The z-component of the total spin operator is:

$$S_z = S_z^{(1)} + S_z^{(2)}$$

$$\uparrow\uparrow, \ \uparrow\downarrow, \ \downarrow\uparrow, \ \downarrow\downarrow$$

$$S_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)})\chi_1 \chi_2 = (S_z^{(1)} \chi_1)\chi_2 + \chi_1 (S_z^{(2)} \chi_2)$$
$$= (\hbar m_1 \chi_1)\chi_2 + \chi_1 (\hbar m_2 \chi_2) = \hbar \underbrace{(m_1 + m_2)}_{m} \chi_1 \chi_2$$

for particle 1, this can be up or down along the z axis

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

$$S_z \chi_1 \chi_2 = \hbar\, m\, \chi_1 \chi_2 \rightarrow$$

$$\uparrow\uparrow: \quad m = 1;$$

$$\uparrow\downarrow: \quad m = 0;$$

$$\downarrow\uparrow: \quad m = 0;$$

$$\downarrow\downarrow: \quad m = -1.$$

What does this mean? Two combinations with $m$=0? Use "lowering operators" (or "raising operators") to find special combinations.

$$S_- = S_-^{(1)} + S_-^{(2)}$$

$$S_-(\uparrow\uparrow) = (S_-^{(1)} \uparrow) \uparrow + \uparrow (S_-^{(2)} \uparrow)$$

$$= (\hbar \downarrow) \uparrow + \uparrow (\hbar \downarrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

Not in book:

$$S_-(\downarrow\uparrow + \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow + \uparrow\downarrow) =$$

$$= \underbrace{(S_-^{(1)}\downarrow)}_{=0}\uparrow + \underbrace{(S_-^{(1)}\uparrow)}_{(\hbar\downarrow)}\downarrow + \downarrow\underbrace{(S_-^{(2)}\uparrow)}_{(\hbar\downarrow)} + \uparrow\underbrace{(S_-^{(2)}\downarrow)}_{=0} =$$

$$= 2\hbar\downarrow\downarrow$$

This means that $\uparrow\uparrow$ , $(\downarrow\uparrow + \uparrow\downarrow)$ , and $\downarrow\downarrow$ are linked forming a set called the triplet.

Make sure you understand this page and next, step by step.

Not in book:

$$S_-(\downarrow\uparrow - \uparrow\downarrow) = (S_-^{(1)} + S_-^{(2)})(\downarrow\uparrow - \uparrow\downarrow) =$$

$$= \underbrace{(S_-^{(1)}\downarrow)}_{=0}\uparrow - \underbrace{(S_-^{(1)}\uparrow)}_{(\hbar\downarrow)}\downarrow + \downarrow\underbrace{(S_-^{(2)}\uparrow)}_{(\hbar\downarrow)} - \uparrow\underbrace{(S_-^{(2)}\downarrow)}_{=0} =$$

$$= 0.$$

If you apply $S_+$ it gives zero as well. This means that $(\uparrow\downarrow - \downarrow\uparrow)$ is alone forming a singlet.

$$\left\{\begin{array}{ll} |1\,1\rangle &= \uparrow\uparrow \\ |1\,0\rangle &= \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,-1\rangle &= \downarrow\downarrow \end{array}\right\} \quad s = 1$$

(triplet; like for *L* angular momentum, projections are discrete)

$$\left\{|0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right\} \quad s = 0$$

(singlet)

$|s\ m\rangle$

*s* denotes the total spin and m denotes the z-axis projection of the total spin.

What we called the "triplet" has three states, but is it truly a state of total spin 1?

Consider the **total** spin operator:

The complicated part

$$S^2 = (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) \cdot (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$$

Example for state "up down":

$$\begin{aligned}\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}(\uparrow\downarrow) &= (S_x^{(1)} \uparrow)(S_x^{(2)} \downarrow) + (S_y^{(1)} \uparrow)(S_y^{(2)} \downarrow) + (S_z^{(1)} \uparrow)(S_z^{(2)} \downarrow) \\ &= \left(\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(\frac{i\hbar}{2}\downarrow\right)\left(\frac{-i\hbar}{2}\uparrow\right) + \left(\frac{\hbar}{2}\uparrow\right)\left(\frac{-\hbar}{2}\downarrow\right) \\ &= \frac{\hbar^2}{4}(2\downarrow\uparrow - \uparrow\downarrow)\end{aligned}$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ i \end{pmatrix} = \frac{\hbar}{2} i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly: $$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}(\downarrow\uparrow) = \frac{\hbar^2}{4}(2\uparrow\downarrow - \downarrow\uparrow)$$

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}|1\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}|1\,0\rangle$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$S^2|1\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|1\,0\rangle = 2\hbar^2|1\,0\rangle$$

Total spin squared

s(s+1) = 1 (1+1) =2

$(S^{(1)})^2$ $\quad$ $(S^{(2)})^2$ $\quad$ $2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$

s(s+1) = 1/2 (1/2 + 1) =3/4

Same story with "singlet" (left as exercise):

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} |0\,0\rangle = \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} (2 \downarrow\uparrow - \uparrow\downarrow - 2 \uparrow\downarrow + \downarrow\uparrow) = -\frac{3\hbar^2}{4} |0\,0\rangle$$

Just this sign difference with previous page!

$$S^2 |0\,0\rangle = \left( \frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - 2\frac{3\hbar^2}{4} \right) |0\,0\rangle = 0$$

s(s+1) = 0 (0+1) =0

Also left as exercise, application of total spin over "up up" $|1\;1\rangle$ and "down down" $|1\;-1\rangle$ .

$|s\ m\rangle$

Then, we have confirmed that indeed we form a **triplet** and a **singlet**, out of two spins ½.

$$\left\{\begin{array}{ll} |1\ 1\rangle & = \uparrow\uparrow \\ |1\ 0\rangle & = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\ -1\rangle & = \downarrow\downarrow \end{array}\right\} \quad s = 1$$

triplet

$$\left\{|0\ 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\right\} \quad s = 0$$

singlet