If two electrons are in the same state, such as (1s,↑), then the "-" wave function cancels:

$$\psi_-(\mathbf{r}_1, \mathbf{r}_2) = A[\psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2) - \psi_a(\mathbf{r}_1)\psi_a(\mathbf{r}_2)] = 0$$

Then, the famous **Pauli principle** can be deduced from the fact that electrons are fermions.

This is somewhat similar to the case when we deduced the **uncertainty principle** in QM411.

It is important to keep the number of arbitrary laws to a minimum!

Let us define the **exchange operator** P.

$$P\,\psi(\mathbf{r}_1, \mathbf{r}_2) = \psi(\mathbf{r}_2, \mathbf{r}_1)$$

Applying P again:

$$P\,P\,\psi(\mathbf{r}_1, \mathbf{r}_2) = P\,\psi(\mathbf{r}_2, \mathbf{r}_1) = \psi(\mathbf{r}_1, \mathbf{r}_2)$$

Because $P^2 = 1$ as operator, then the eigenvalues are +1 and -1.

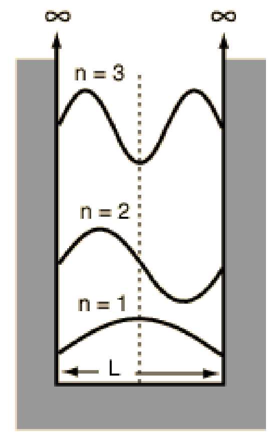
Also $[H,P]=0$:

$$H = \left\{ -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{r_1} \right\} + \left\{ -\frac{\hbar^2}{2m}\nabla_2^2 - \frac{1}{4\pi\epsilon_0}\frac{2e^2}{r_2} \right\} + \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Example 5: Consider two particles without spin in the 1D infinite square well

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \qquad E_n = n^2 K$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

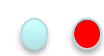


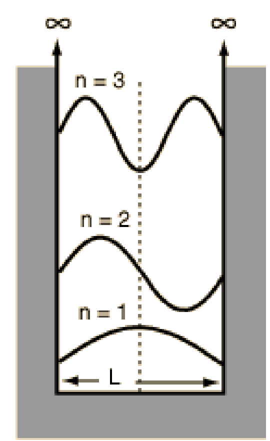
x = 0 at left wall of box.

(1) If particles are distinguishable, then

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2), \qquad E_{n_1 n_2} = (n_1^2 + n_2^2)K$$

For ground state $n_1 = n_2 = 1$:

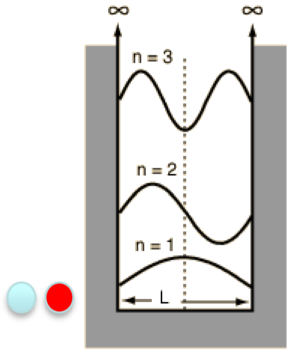




x = 0 at left wall of box.

Ground state $n_1 = n_2 = 1$:

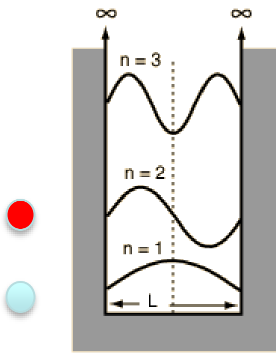
$$\psi_{11} = \frac{2}{a}\sin(\pi x_1/a)\sin(\pi x_2/a), \qquad E_{11} = 2K$$



x = 0 at left wall of box.

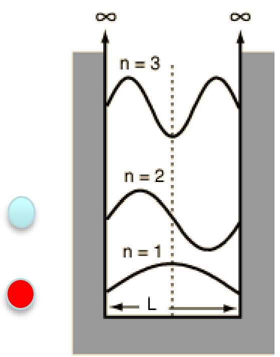
First excited state $n_1 = 1$, $n_2 = 2$ or $n_1 = 2$, $n_2 = 1$

$$\psi_{12} = \frac{2}{a}\sin(\pi x_1/a)\sin(2\pi x_2/a), \qquad E_{12} = 5K$$



x = 0 at left wall of box.

$$\psi_{21} = \frac{2}{a}\sin(2\pi x_1/a)\sin(\pi x_2/a), \qquad E_{21} = 5K$$

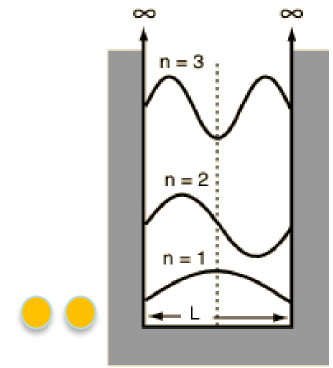


x = 0 at left wall of box.

Degeneracy = 2

(2) If particles are bosons:

Ground state (E=2K) is the same $n_1 = n_2 = 1$:



x = 0 at left wall of box.

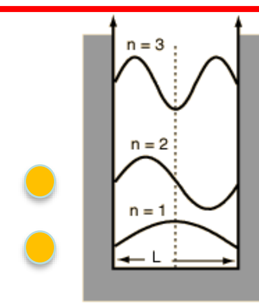
$$\psi_{11} = \frac{2}{a}\sin(\pi x_1/a)\sin(\pi x_2/a), \qquad E_{11} = 2K$$

$$P\,\psi(x_1,x_2) = \psi(x_2,x_1) = +\psi(x_1,x_2)$$

First excited state (E=5K) is now nondegenerate

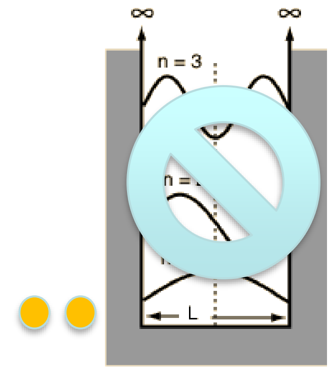
$$\psi^{\text{excited}}(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin(\pi x_1/a)\sin(2\pi x_2/a) + \sin(2\pi x_1/a)\sin(\pi x_2/a)\right]$$

$$P\,\psi^{\text{excited}}(x_1,x_2) = \psi^{\text{excited}}(x_2,x_1) = +\psi^{\text{excited}}(x_1,x_2)$$



x = 0 at left wall of box.

(3) If particles are fermions:



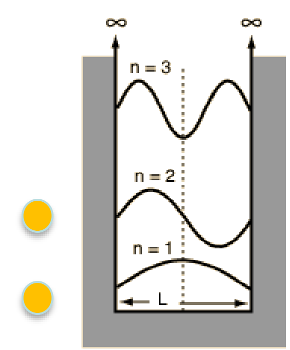
x = 0 at left wall of box.

Because temporarily I am forgetting about the spin, then I cannot place 2 particles in $n_1 = 1$.

Then, the true ground state (E=5K) becomes:

$$\psi(x_1,x_2) = \frac{\sqrt{2}}{a}\left[\sin(\pi x_1/a)\sin(2\pi x_2/a) - \sin(2\pi x_1/a)\sin(\pi x_2/a)\right]$$

$$P\,\psi(x_1,x_2) = \psi(x_2,x_1) = -\psi(x_1,x_2)$$



x = 0 at left wall of box.

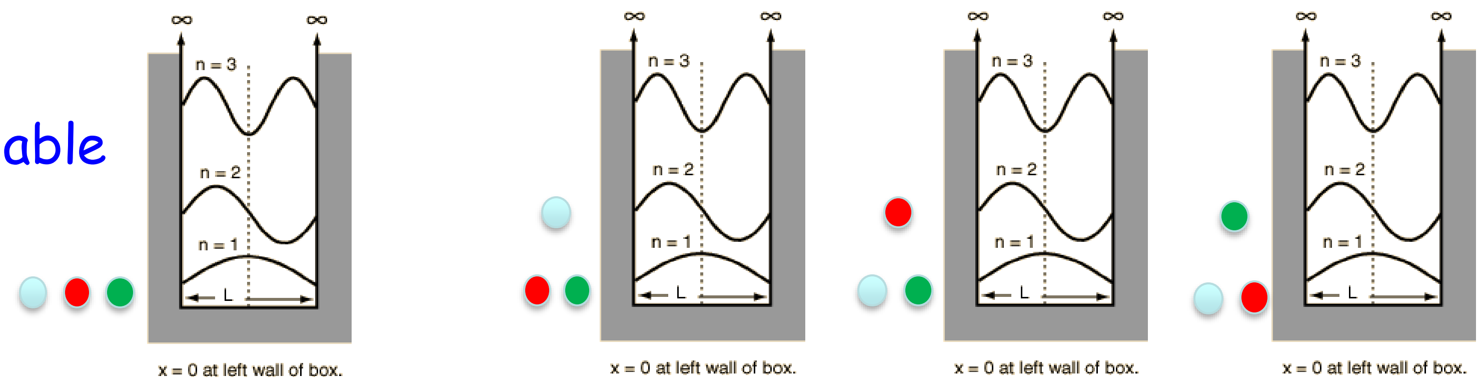
In summary:

	distinguishable	bosons	fermions
E=2K	1	1	0
E=5K	2	1	1
etc			

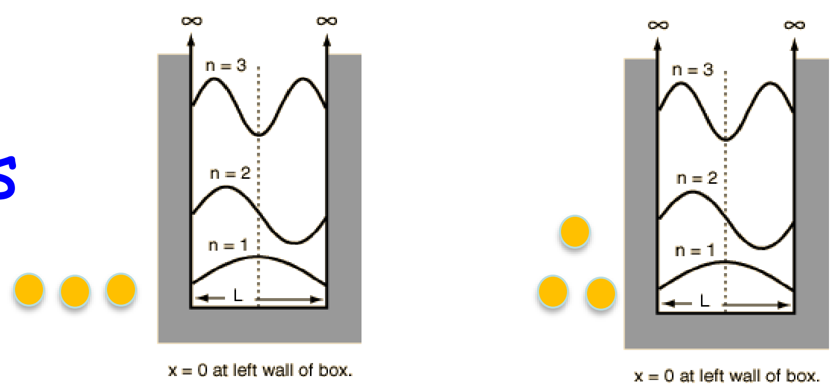
This has profound implications for many-body physics and for thermodynamics

Three particles?

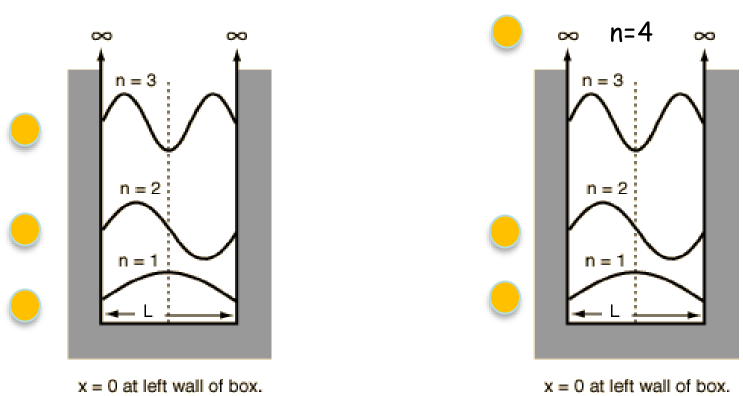
distinguishable



bosons



fermions



5.1.2 Exchange Forces

This is an "effective force", not a real one.

It is another of the consequences of having symmetric or antisymmetric combinations.
Consider 1D and states a and b.

distinguishable

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$$

bosons

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2)]$$

fermions

$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2)]$$

Consider the typical "distance" between particles, via the quantity:

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

Distance in 1D

distinguishable

$$\langle x_1^2 \rangle = \int x_1^2 |\psi_a(x_1)|^2 \, dx_1 \int |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \int |\psi_a(x_1)|^2 \, dx_1 \int x_2^2 |\psi_b(x_2)|^2 \, dx_2 = \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 \, dx_1 \int x_2 |\psi_b(x_2)|^2 \, dx_2 = \langle x \rangle_a \langle x \rangle_b$$

$$\langle (x_1 - x_2)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

The main punchline is that the distance changes for the symmetric and antisymmetric combinations:

$$\langle (x_1 - x_2)^2 \rangle_\pm = \underbrace{\langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b}_{\text{distinguishable}} \mp 2|\langle x \rangle_{ab}|^2$$

bosons (upper sign), fermions (lower sign)

So even non-interacting particles (no Coulomb repulsion at all) suffer an effective force: bosons attract and fermions repel.