Adding both terms appears to lead to a long expression

$$\boxed{E_r^1 = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right]} + \boxed{E_{so}^1 = \frac{(E_n)^2}{mc^2}\left\{\frac{n[j(j+1) - l(l+1) - 3/4]}{l(l+1/2)(l+1)}\right\}}$$

Consider j=l+1/2 first. Then j(j+1) = (l+1/2)(l+3/2). Careful algebra (HW problem) leads to

$$E_{fs}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

Consider now j=l-1/2. Then j(j+1) = (l-1/2)(l+1/2). Careful algebra (HW problem) leads to the **same**

$$E_{fs}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

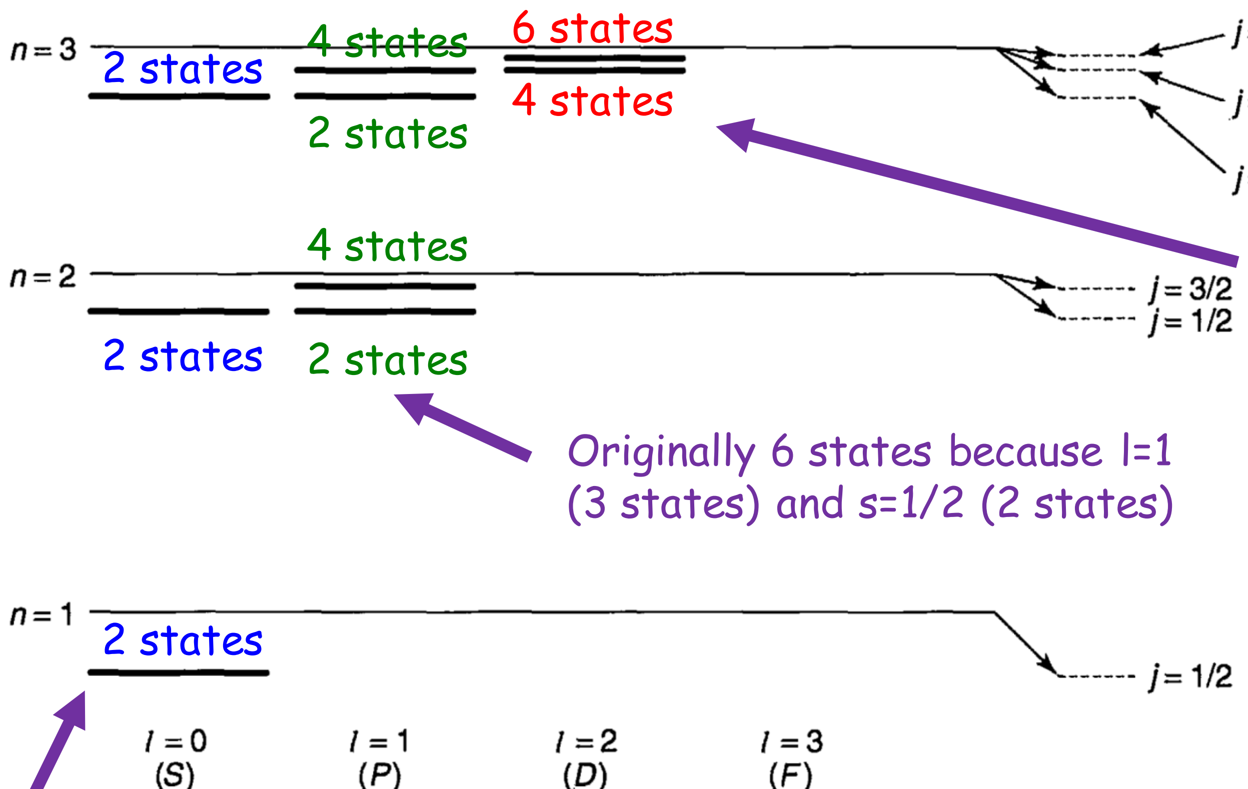
In HW problem you will show that order 0 + order 1 gives:

$$\boxed{E_{nj} = -\frac{13.6\,\text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]}$$

where

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \cong \frac{1}{137.036}$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$



Originally 10 states because l=2 (5 states) and s=1/2 (2 states)

Originally 6 states because l=1 (3 states) and s=1/2 (2 states)

Originally 2 states because l=0 (1 state) and s=1/2 (2 states).

NOTE: this is for the H atom. For He and beyond the e-e repulsion plays a more important role than the fine structure.

Counting of states again:

If we consider the “p levels”, they have angular momentum $l=1$. The spin is $s=1/2$. Then, the total angular momentum j can be 1+1/2=3/2 and 1-1/2=1/2 .

The original $l=1$ is 3 states $m_l = 1,0,-1$. The original spin $s=1/2$ is 2 states $m_s = 1/2, -1/2$. Total = 6 states.

The $j=3/2$ is 4 states $m_j = 3/2, 1/2, -1/2, -3/2$. The $j=1/2$ is 2 states $m_j = 1/2, -1/2$. Total = 6 states. As expected, the original 6 degenerate states are simply mixed in a linear combination to become states with j sharply defined.

6.5 Hyperfine Splitting:

These are corrections to the H atom that are even much smaller than the fine structure corrections, yet they are very important.

$$\boldsymbol{\mu}_e = -\frac{e}{m_e}\mathbf{S}_e \qquad \boldsymbol{\mu}_p = \frac{g_p e}{2m_p}\mathbf{S}_p$$

The proton also has a spin, like the electron. The magnitude of the dipole moment is much smaller because of m_p in the denominator. Still μ_p is nonzero. g_p is ~5 because the proton is composed of 3 quarks.

Any dipole moment is felt like a magnetic field by the other spin, thus an interaction arises which for the l=0 ground state is:

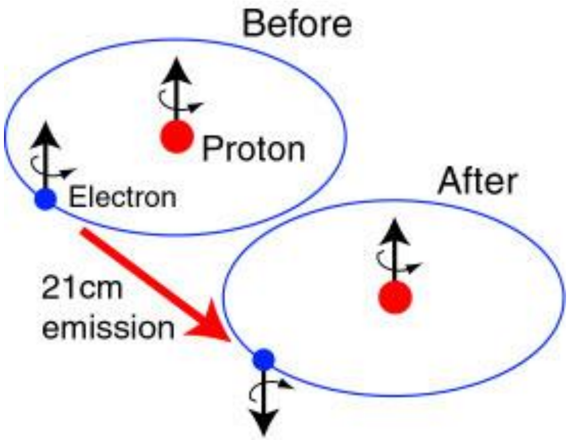
$$E^1_{\text{hf}} = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3}\langle \mathbf{S}_p \cdot \mathbf{S}_e \rangle$$

Similar to: $\mathbf{S}\cdot\mathbf{L}$

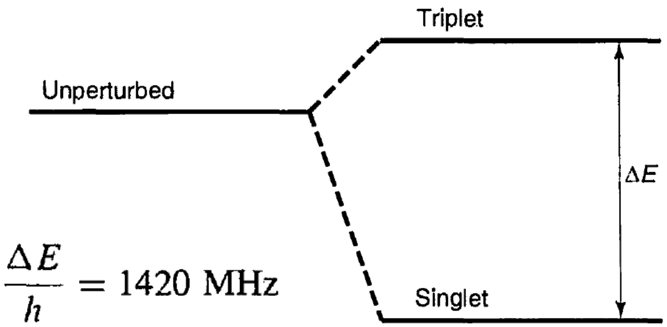
$$E^1_{\text{hf}} = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4}\begin{cases} +1/4, & \text{(triplet)};\\ -3/4, & \text{(singlet)}.\end{cases}$$

$$\boxed{\Delta E = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4} = 5.88\times 10^{-6}\ \text{eV}}$$

NOTE: In HW you will show that the correction coming from the finite size of the nucleus is even smaller.



Spin flip transition

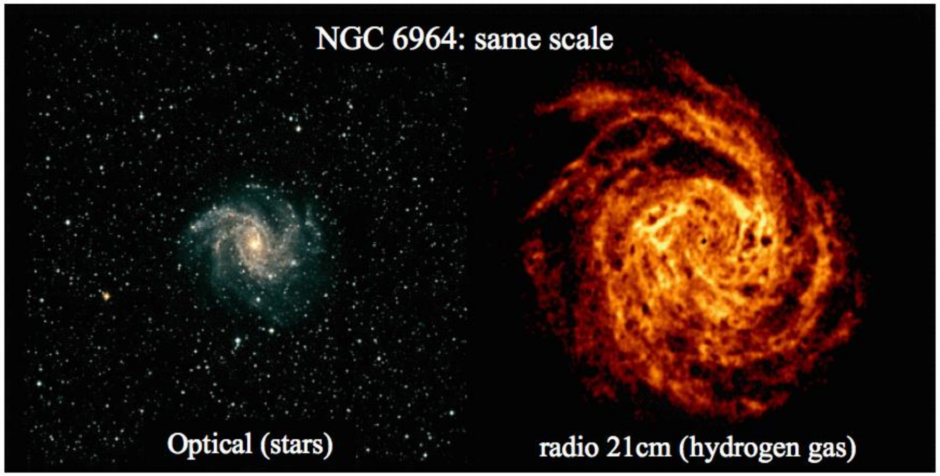


$$\nu = \frac{\Delta E}{h} = 1420 \text{ MHz}$$

$$c/\nu = 21 \text{ cm}$$

Lifetime 10M years.

The 21-cm line is a “fingerprint” of hydrogen gas, which is 90% of the gas between stars. The emitted radiation penetrates “dust” in space, that blocks visible light. The spiral arms of galaxies, made mostly of H, can be mapped studying this frequency.



Galaxy about 1/3 size Milky Way.