Anderson Localization

Andrew Nicholson[1]

[1] *Department of Physics and Astronomy, The University of Tennessee, Knoxville, Tennessee 37996, USA*

(Dated: May 13, 2009)

I. INTRODUCTION

From the start of their studies students in condensed matter physics are exposed to ordered systems. These systems can be very simple and give insight into the physics of many materials. However it seems that much of what is learned in ordered systems cannot be brought to disordered systems. Instead new conceptual understandings much be reached will very different consquences. One of these is the absence of movement of charge carriers in strongly disordered systems. The reason for this has been coined Anderson Localization. In this paper ordered systems will be breifly reviewed moving on to a conceptual picture of localization. Then region between perfect order and strong disorder will be investegated using scaling theory. Then we will look into connecting these two regimes using scaling theory.

II. ORDERED SYSTEMS

Many areas of present and past research in condensed matter physics focuses on ordered systems. These systems are easy to study by use of very powerfull theorems which produce simple results. For example perfectly regular crystalline lattices are used to describe different materials. In many of these materials the free electron gas approximation works very well. In this approximation an electron of mass m is confined in an infinite square well of length l. The result is that electron wave functions are described as planewaves running throughout the crystal.

$$\Psi(r) = e^{i(k\cdot r)} \tag{1}$$

with the simple band structure given by:

$$\epsilon = \frac{\hbar^2}{2m}(k^2) \tag{2}$$

The importance of these simple assumptions and results cannot be understated. It is amazing that it works so well. To make this picture more realistic a potential is needed. In the next section a weak peroidic potential is added.

A. Periodic Potential and Bloch's Theorem

In a weak periodic potential we can use much of what we learned before and apply Bloch's Theorem. In this approximation a periodic potential, formed by the ions in a lattice, are approximated by an regular array of potential wells. Using Bloch's Theorem the wave functions assume of the form of planewaves with a function of the periodicity of the lattice:

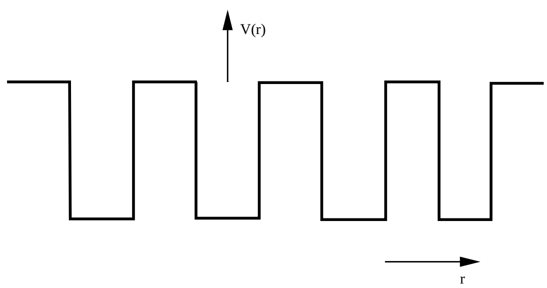

FIG. 1: Periodic Potential approximated as a regular set of square wells[8]

$$\psi_i(r) = exp[ik \cdot r] f_i(r) \tag{3}$$

where

$$f_i(r) = \sum_G c_{i,G} exp[iG \cdot r] \tag{4}$$

resulting in

$$\psi_i(r) = \sum_G c_{i,k+G} exp[i(k+G) \cdot r] \tag{5}$$

where G is the reciprocal lattice vectors. These reciprocal lattice vectors are defined by $G \cdot l = 2\pi m$ for all l where l is a lattice vector of the crystal and m is an integer. In the weak periodic potential the coefficents c. An important thing to note is that electrons are freely propogating in this model. Here we can see the electron wave functions move throughout the crystal giving us the normal electrical conductivity.

B. Conductivity

In this picture resistance is created in metals via scattering off impurities. Refering to the Boltzmann transport equation the temperature dependence of conductivity is

$$\sigma(T) = \sigma_0 - A\sigma_0^2 T^n \tag{6}$$

where σ_0 is the residual conductivity due to impurity scattering. As temperature increases the ammount of scattering increases thus the conductivity decreases meaning that $A > 0$ and n is a positive interger. However in systems with strong disorder this breaks down! A can be positive or negative and n is usually 1/2. This leads us to believe that we will have to rethink the way we describe systems with little order.

III. STRONG DISORDER

In a strongly disordered system we have to rethink our approach of solving problems. Instead of a perfect periodic potential consider a strongly varying potential. Unlike our picture before the size of wells are far from uniform. Moreover if we treat hybridization between different wells as a pertubation an extended state becomes unlikley. Wavefunctions which have the most overlap will exist in near neighbor wells however the energy difference between these will be large. Thus the energy denominator in the pertubation will become large making movement unlikely. On the other hand, two wells will similar energies will be far apart such that the overlap in their wavefunctions is small again making movement unlikely. This is called Anderson localization.

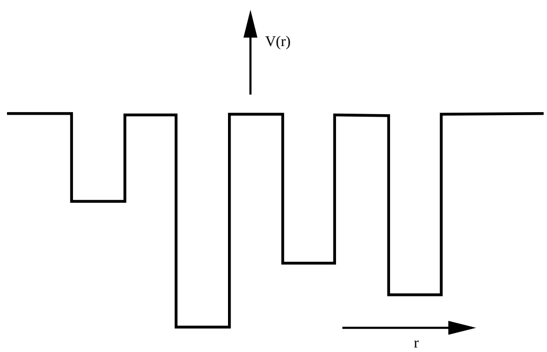

FIG. 2: Strongly varying potential wells[8]

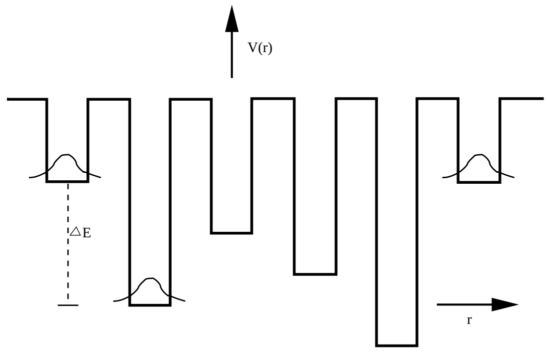

FIG. 3: Electron wave functions depicted in a system in a random potential

A. Localization

In his landmark 1958 paper, Anderson showed that an electron's wave function will differ greatly from that in an ordered system. If electrons become localized as we discussed before then the wavefunction no longer looks like a planewave but decays exponentially like

$$|\Phi(r)| = e^{|r|/\xi} \tag{7}$$

where ξ is the localization length.

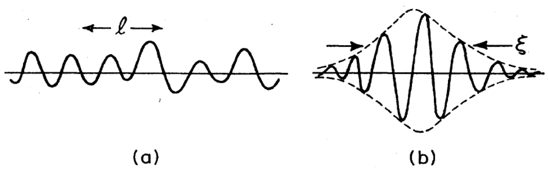

FIG. 4: Wavefunctions of (a) extended state with mean free path l (b) localaized state with localization length ξ^2

This understanding of localization is very qualtitative with little attention to the ammount of disorder needed for our argument to become valid. The rest of this paper will be devoted to the regime between no disorder and strong disorder, what will be labeled intermediate disorder. Results from scaleing theory will be dicussed to investegate how dimensionality and size of a lattice contributes to localization.

B. Experimental Verification

Hu et. al. found sound localization in an three dimensional elastic network. In many experiments using classical waves absorbtion of the material has led to problems in investegating Anderson Localization. Hu et. al. used different ultra sound frequencies and measured the intensities of these waves at different points in the network.

As the sound frequency was changed the effect of localization changes. As you can see for 0.2 MHz ultrasound waves the waves have a diffuse character. As the frequency is increased to 2.4 MHz the ultrasound waves become localized. This gives us a sense that the the scale size changes the systems sensitivity to disorder.

IV. INTERMEDIATE DISORDER

We have discussed the two extremes of the problem and now we will investegate the large intermediate region between the two. Scaling theory was developed by Thouless and others to describe this regime. Scaling theory assumes that a good measure of disorder is conductance, G, (not conductivity) and that conductance depends only on scale size, L. It is usefull to use a dimensionless conductance g

$$g = \frac{G}{e^2/\hbar} \tag{8}$$

FIG. 5: Brazed aluminium beads ≈ 4.011 nm in diameter[7]

FIG. 6: Close up look at the brazed aluminium beads[7]

Conductance is a good quantity to use since it is physically measureable and is directly related to the behavior of the system as it doubles in size. From conductance one can build a scaling function $\beta(g)$ which gives one a sense whether states will be localized or not.

A. Scaling Theory

Scaling theory attempts to understand the behavior of conductance g as a function of the size of the system L. From this the scaling function can be defined as

$$\beta(g) = \frac{d(lng)}{d(lnL)} \tag{9}$$

The distance over which the electron wave function changes by a period defines the mean free path l. l is a length scale of interest in the study of localization. At this scale let the conductance be defined as g_0. If disorder is large then g_0 is small and vice versa. It turns out that the conductance has two different asymptotic forms for $L >> l$ depending on the ammount of disorder.

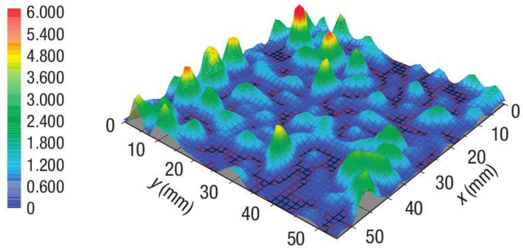

FIG. 7: Normalized intensity throughout the elastic network with 0.2 MHz ultrasound waves.[7]

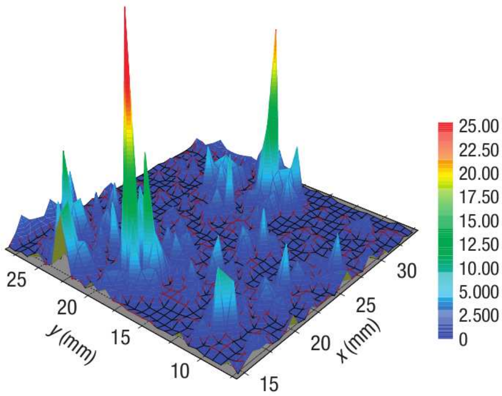

FIG. 8: Normalized intensity throughout the elastic network with 2.4 MHz ultrasound waves.[7]

In the case of weak disorder where the electron wave function is very much like a planewave the conductivity follows conventional conductivity. Electron transport is freely propogating and Ohm's law is valid.

$$\sigma = \frac{nle^2}{\hbar k_F} \tag{10}$$

where n is the electron density. In this case the conductance follows:

$$g(L) = \sigma L^{d-2} \tag{11}$$

On the other hand if localization exists then electron transport is not free propogating but diffusive. Here the relavent scale is the localization length ξ which is larger than l. In this case the conductance takes the form:

$$g(L) \propto e^{-L/\xi} \tag{12}$$

It is important to note that $g(L)$ moves from g_0 to either of these regemes smoothly. The method to the madness is to see how the macroscopic disorder g depends on the microscopic behavior g_0 at scale l and dimension d. One starts at the microscopic scale then expands to a larger scale with the idea that the change in disorder is influenced by its value at a smaller scale. Now we will look into the scaleing function $\beta(g)$ and what it tells us about our two regimes. There exists a characteristic dimensionless conductance g_c which is on the order of π^{-2}. We will use this in our limiting cases.

For $g >> g_c$ we are in the weak disorder regime. In this case our scaling function takes the form:

$$\beta(g) = (d-2) \tag{13}$$

Where d is the number of dimensions. Notice that for $d = 2$ the scaling function is zero.

For $g << g_c$ we are in the strongly disordered regime. In this case the scaling function takes the form

$$\beta(g) = ln(g/g_c) \tag{14}$$

which is independent of dimensionality. Also $\beta(g)$ is negative meaning that that the conductance decreases as the length scale increases.

Between these two regimes is a perturbative regime for large g.

$$\beta(g) = (d-2) - a/g \tag{15}$$

where for an electron gas $a = g_c = /pi^{-2}$. From this we can see that $\beta(g)$ is always less than the perfectly ordered regime. Thus equation (10) is an upper bound on the increase of conductance with scale size or conductance always increases more slowly than implied by equation (10).

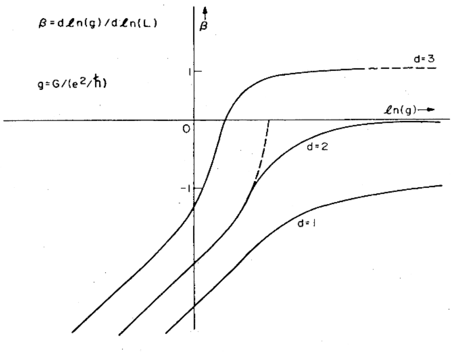

FIG. 9: The scaling function $\beta(g)$ vs g for different dimensions[2]

B. Three Dimensional Localization

In three dimensions $\beta(g)$ has a value of unity for large conductance (no disorder). As conductance decreases (disorder increases) it becomes negative. Let the contannce where $\beta(g) = 0$ be defined as g_3. Suppose now that the conductance at the microscopic cuttoff g_0 at l is larger than g_3. In this case one starts at the positive part of the $\beta(g)$ curve. A small increase in the length scale from l will increase the conductance moving up the $\beta(g)$ curve. Thus at larger and larger scales $\beta(g)$ reaches the unity limit and the system is in the ordered limit. In this case the material acts like a metal. However, if the microscopic cuttoff g_0 at l is smaller than g_3 then $\beta(g)$ is negative. Increasing the scale factor L from l will now decrease the conductance moving down the $\beta(g)$ curve. In this case at large enough scales the system reaches the strongly disordered limit or the localized limit. In this case the material acts like an insulator. If the microscopic conductance g_0 at length scale l is equal to g_3 then we have critical disorder. The fixed point $\beta(g) = 0$ is an unstable fixed point meaning that small deviations lead to much different asymptotic results for increase in scaling.

C. Two Dimensional Localization

In two dimensions $\beta(g) < 0$ for all conductance. This means as the length scale increases only localized behavior is possible. If a system begins with large conductance g_0 at length scale l and the length scale is increased then one moves down the $\beta(g)$ curve until the stronly disordered limit is reached. From this one can infer that there are no truly extended states in two dimensions. All states are localized for the smallest microscopic disorder.

We can approximate the localization length by using the $\beta(g)$ function in the pertubative regime

$$\frac{dlng}{dlnL} = \frac{a}{g} \tag{16}$$

Integrating the above expression between length scales l and L one gets

$$g(L) = g_0 - \frac{e^2}{\hbar\pi^2} ln(\frac{L}{l}) \tag{17}$$

In conventional transport theory $g_0 = \frac{e^2}{2\pi\hbar}k_F l$ so that for $L = \xi$

$$\xi = le^{(\frac{k\pi^2 g_0}{e^2})} = le^{(\frac{\pi}{2}k_F l)} \tag{18}$$

From this we can see that the localization length depends exponentially on the mean free path l.

D. One and Two Dimensional Localization

In one dimension $\beta(g) < 0$ for all g thus the slightest introduction of disorder creates localization. The two dimensional problem is essentially the same but more interesting. Until the forumlation of scaling theory it was assumed that there was some critical conductance for two dimensional systems (the dotted line in the scaling function plot). Scaleing theory showed that this was wrong. If a two dimensional system at some scale L has a large conductance (low disorder limite) is increased in size we will move down the β curve until we move to the stongly disordered limit.

E. Temperature Dependence

All of the above has been for zero temperature, but what about when $T > 0$? The answer depends on the dimensionality. For $3D$

$$\sigma_{3D} = \sigma_0 + \frac{e^2}{\hbar\pi^3}\frac{1}{a}T^{p/2} \tag{19}$$

for $2D$:

$$\sigma_{2D} = \sigma_0 + \frac{p}{2}\frac{e^2}{\hbar\pi^3}ln(\frac{T}{T_0}) \tag{20}$$

and last for $1D$

$$\sigma_{1D} = \sigma_0 - \frac{ae^2}{\hbar\pi^3}T^{-p/2} \tag{21}$$

where p is the scattering index depending on the scattering mechanism, dimensionallity, etc. As we can see from these conductivity decreaeses with decreasing temperature. We can think about this in this way: as the temperature decreases the effective scale size increaseses so that localization becomes more pronounced. The first experimental confirmation of $/sigma_{2D}$ was done in 1979 by Dolan and Osheroff[9]. A way of making sense of this is that increasing temperature may create excited states in the strongly varying potential wells making the tunneling much more probable.

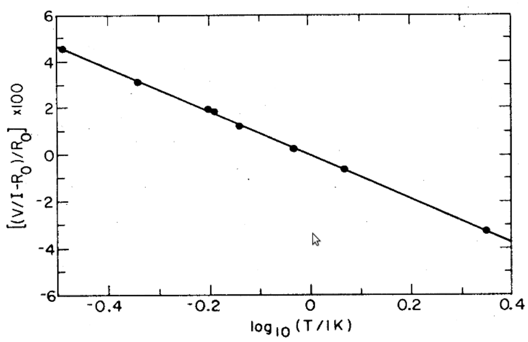

FIG. 10: Resistivity vs. lnT for PdAu film[9]

V. CONCLUTION

We have seen how disordered systems act very differently from their ordered counterparts. While electrons in ordered systems act as Bloch waves which are diffusive throughout the material, electrons in stronly disordered systems are diffusive. They move from one strongly varying potential well with a very small tunneling amplitude. In scaleing theory we used conductance as a dimensionless measure of disorder and have seen how scale size can change the properties of a disordered system. A three dimensional system may reach the limit of weak disorder if at some smaller scale the system is sufficiently ordered (ordered enough to have a conductance above the critical conductance). If not it is doomed to lose conductance as scale size increases. The two dimensional system must have perfect order to avoid localization; the smallest ammount of disorder will enforce localized behavior at larger and larger scales. A one dimensional system has no choice but to display localization properties. Like scale, temperature also has an effect on conductance. Increasing the temperature will increase conductance making it very important to experimentalists to use low temperatures to investigate localization.

---

[1] P. W. Anderson, Phys. Rev., 109, 1492 (1958)
[2] P. Lee, T. V. Ramakrishnan Rev. Mod. Phys., 57, 287 (1985)
[3] P. W. Anderson, Phys. Rev., 124, 41 (1961)
[4] P. W. Anderson, Science, 177, 4047 (1972)
[5] D. S. Wiersma et al, Nature, 390, 671 (1997)
[6] N. W. Ashcroft, N. D. Mermin *Solid State Physics*, 1976, (Holt Saunders, Philadelphia) p. 113
[7] H. Hu *et al*, Nature Physics, 4, 945 (2008)
[8] N. F. Mott E. A. Davis *Electronic Processes in Non-Crystalline Materials*, 1979, (Clarendon, Oxford) p. 16
[9] Dolan, G. J., and D. D. Osheroff, 1979, Phys. Rev. Lett. 43, 721.