

This is a rigorous mathematical proof that the "correction factor" " $\frac{1}{1-\frac{v}{c}}$ " discussed in Griffiths, via intuitive arguments, is indeed correct.

Consider the scalar potential φ as a function of \vec{r} , including the retardation time. To be more specific:

$$\varphi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{p(\vec{r}', t_r) d^3 r'}{|\vec{r} - \vec{r}'|}$$

with $t_r = t - \frac{1}{c} |\vec{r} - \vec{r}'|$. This formula was deduced in class, and it is also well explained and deduced in Jackson's and Griffith's book.

There is a similar formula for $\vec{A}(\vec{r}, t)$, and the steps below to get " $\frac{1}{1-\frac{v}{c}}$ " are similar to those to be followed for φ and \vec{A} . Thus, only φ will be done here.

Consider now directly the case of a point like charge i.e.

$$\varphi(\vec{r}', t_r) = q \delta(\vec{r}' - \vec{w}(t_r))$$

The δ -function tells you it's a point charge. In Griffiths, the "point" is reached via a limit of a finite size ball. In here, we directly study a point charge.

Note that $\vec{w}(t)$ is the fixed trajectory of the particle, given to you. For example it could be a straight line with the particle moving at a constant velocity. In this example $\vec{w}(t) = \vec{v} \cdot t$. But it could be some other trajectory, of course.

Thus far,

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\vec{r}' - \vec{w}(t_r)) d^3 r'}{|\vec{r} - \vec{r}'|}$$

Note that the "argument" of the δ -function, i.e. what is in parenthesis, is complex since t_r depends on \vec{r}' . I.e.

$\delta(\vec{r}' - \vec{w}(t_r)) = \delta(\vec{r}' - \vec{w}(t - \frac{1}{c} |\vec{r} - \vec{r}'|))$. For example, if the velocity is constant, then $\vec{w}(t_r) = \vec{v} t_r$ and

$$\begin{aligned} \delta(\vec{r}' - \vec{w}(t_r)) &= \delta(\vec{r}' - \vec{v}(t - \frac{1}{c} |\vec{r} - \vec{r}'|)) = \delta(\vec{r}' - \vec{r} + \frac{\vec{v}}{c} |\vec{r} - \vec{r}'|) \\ &= \delta(\vec{r}' - \vec{r} t + \frac{\vec{v}}{c} \sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + \vec{r}'^2}). \text{ Clearly complex!} \end{aligned}$$

3.

" ↗ i.e. complicated
" ↗ (not as complex number)

Mathematically, having a complex argument in a δ -function is difficult to handle. Actually, this precise issue is what will produce the " $\frac{1}{t - \gamma_c}$ " correction at the "end of the day"!

Moreover, the argument of the δ -function is a vector! We do not have easy formulas for complex functions that are vectorial, when inside the δ -function.

Thus, we will introduce a new dummy variable

\tilde{t}_r vs

$$\int d\tilde{t}_r \delta(\tilde{t}_r - t_r) = 1$$

i.e. t_r has all the complex \vec{r}' , etc., dependence while \tilde{t}_r is just a variable of integration. Then,

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\vec{r}' - \vec{w}(\tilde{t}_r))}{|\vec{r}' - \vec{r}|} \left(\int \delta(\tilde{t}_r - t_r) d\tilde{t}_r d^3 r' \right)$$

↑ ↑
 dummy still
 variable $t - \frac{1}{c} |\vec{r} - \vec{r}'|$,
 as usual.

The key issue now is to integrate first in $d^3 r'$:

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\tilde{t}_r \frac{\delta(\tilde{t}_r - (t - \frac{1}{c} |\vec{r} - \vec{w}(\tilde{t}_r)|))}{|\vec{r} - \vec{w}(\tilde{t}_r)|} =$$

$$= \frac{q}{4\pi\epsilon_0} \int d\tilde{t}_r \frac{\delta(\tilde{t}_r - t + \frac{1}{c} |\vec{r} - \vec{w}(\tilde{t}_r)|)}{|\vec{r} - \vec{w}(\tilde{t}_r)|}$$

There is no longer a vector in the argument of the δ -function, but still the function in parentheses is complex. For example, if $\vec{w} = \vec{w}(t_r) = \vec{v} t_r$, then (i.e. complicated)

$$\begin{aligned} \delta(\dots) &= \delta(\tilde{t}_r - t + \frac{1}{c} |\vec{r} - \vec{v} \tilde{t}_r|) = \\ &= \delta(\tilde{t}_r - t + \frac{1}{c} \sqrt{r^2 - 2\vec{r} \cdot \vec{v} \tilde{t}_r + v^2 \tilde{t}_r^2}) \end{aligned}$$

In general, $\tilde{t}_r - t + \frac{1}{c} (\vec{r} - \vec{w}(\tilde{t}_r))$ is a complex (complicated) function $f(\tilde{t}_r)$.

So, a fundamental mathematical step for all the rest is the following: how do we handle a

$$\delta(f(\tilde{t}_r)) ?$$

There is a crucial math formula that says that

$$S[f(\tilde{t}_r)] = \sum_i \frac{\delta(\tilde{t}_r - t_i)}{\left| \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} \right|} \quad \begin{pmatrix} t_i = \text{zeros of} \\ \text{the equation} \\ f(\tilde{t}_r) = 0 \end{pmatrix}$$

$\sim_{\text{at}} \tilde{t}_r = t_i$

For example,

$$\delta(x^2 - a^2) = \frac{1}{2|a|} [\delta(x-a) + \delta(x+a)] \quad (a \neq 0)$$

For more details, please check literature addressing properties of δ -functions. In our case,

$$f(\tilde{t}_r) = \tilde{t}_r - t + \frac{1}{c} |\vec{r} - \vec{w}(\tilde{t}_r)|.$$

For example, if $\vec{w}(\tilde{t}_r) = \vec{v} \tilde{t}_r$ then example 10.3 of Griffiths shows that the equation of $f(\tilde{t}_r) = 0$ is simply a quadratic:

$$\tilde{t}_r - t + \frac{1}{c} \sqrt{\vec{r}^2 - 2\vec{r} \cdot \vec{v} \tilde{t}_r + v^2 \tilde{t}_r^2} = 0$$

There are 2 solutions, $\tilde{t}_r = \frac{-\pm\sqrt{\dots}}{\dots}$ but

from physical arguments Griffiths shows that only one of the signs \pm is acceptable. So there is only 1 solution.

This is a general property: for any given space-time point (\vec{r}, t) and source trajectory \vec{w} there is only one retarded time t_r .

Then,

$$\delta(f(\tilde{t}_r)) = \frac{\delta(\tilde{t}_r - t_r)}{\left| \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} \right|_{\tilde{t}_r=t_r}}$$

where t_r is the only acceptable (physically) solution of the equation

$$\tilde{t}_r - t + \frac{1}{c} |\vec{r} - \vec{w}(\tilde{t}_r)| = 0$$

Note that before, when we introduced

$\int \delta(\tilde{t}_r - t_r) d\tilde{t}_r = 1$,
that t_r was $t - \frac{1}{c} |\vec{r} - \vec{r}'|$ i.e. \vec{r}' was not yet $\vec{w}(t_r)$.
The t_r now above in this page is $t - \frac{1}{c} |\vec{r} - \vec{w}(t_r)|$

then, so far we have:

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\tilde{t}_r \frac{\delta(\tilde{t}_r - t_r)}{|\vec{r} - \vec{w}(\tilde{t}_r)|} \cdot \frac{1}{\left| \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} \right|_{\tilde{t}_r=t_r}}$$

$$\text{where } t_r = t - \frac{1}{c} |\vec{r} - \vec{w}(t_r)|$$

Let us calculate $\frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r}$:

$$\begin{aligned}
 \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} &= \frac{2}{\partial \tilde{t}_r} \left(\tilde{t}_r - t + \frac{1}{c} |\vec{r} - \vec{w}(\tilde{t}_r)| \right) = \\
 &= \frac{2}{\partial \tilde{t}_r} \left(\tilde{t}_r - t + \frac{1}{c} \sqrt{[\vec{r} - \vec{w}(\tilde{t}_r)] \cdot [\vec{r} - \vec{w}(\tilde{t}_r)]} \right) = \\
 &= 1 - 0 + \frac{1}{c} \frac{2}{\partial \tilde{t}_r} \sqrt{(\vec{r} - \vec{w}(\tilde{t}_r)) \cdot (\vec{r} - \vec{w}(\tilde{t}_r))} \\
 &= 1 + \frac{1}{c} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{[(\vec{r} - \vec{w}(\tilde{t}_r)) \cdot (\vec{r} - \vec{w}(\tilde{t}_r))]}} \frac{2}{\partial \tilde{t}_r} (\vec{r} - \vec{w}(\tilde{t}_r)) \cdot (\vec{r} - \vec{w}(\tilde{t}_r)) \\
 &\quad = \vec{n}(\tilde{t}_r) \\
 &\quad \text{by definition} \\
 &= 1 + \frac{1}{2c |\vec{r}|} \cdot \frac{2}{\partial \tilde{t}_r} \vec{n}(\tilde{t}_r) \cdot \vec{n}(\tilde{t}_r)
 \end{aligned}$$

In general, $\frac{\partial}{\partial \tilde{t}_r} \vec{a}(\tilde{t}_r) \cdot \vec{a}(\tilde{t}_r) = \frac{\partial}{\partial \tilde{t}_r} (a_x^2 + a_y^2 + a_z^2) =$

$$\begin{aligned}
 &= 2a_x \frac{\partial a_x}{\partial \tilde{t}_r} + 2a_y \frac{\partial a_y}{\partial \tilde{t}_r} + 2a_z \frac{\partial a_z}{\partial \tilde{t}_r} = 2 \vec{a} \cdot \frac{\partial \vec{a}}{\partial \tilde{t}_r}
 \end{aligned}$$

for an arbitrary \vec{a}

Then,

$$\frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} = 1 + \frac{1}{c|\vec{r}|} \cancel{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial \tilde{t}_r}$$

$$\text{where } \vec{r} = \vec{r} - \vec{w}(t_r)$$

$$\frac{\partial \vec{r}}{\partial \tilde{t}_r} = -\frac{\partial \vec{w}(t_r)}{\partial t_r} \stackrel{\uparrow}{=} -\vec{v}(t_r)$$

by definition

$$\frac{\partial \vec{w}}{\partial t} = \vec{v}(t)$$

velocity of
particle along the
trajectory.

$$\begin{aligned} \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} &= 1 - \frac{1}{c} \cdot \frac{\vec{r} \cdot \vec{v}}{|\vec{r}|} = 1 - \frac{1}{c} \hat{\vec{r}} \cdot \vec{v} \\ &\equiv \frac{\vec{r} - \vec{w}(t_r)}{|\vec{r} - \vec{w}(t_r)|} \stackrel{\uparrow}{=} \vec{v}(t_r) \end{aligned}$$

We have arrived to the " $\frac{1}{1-v}$ " correction factor! It came from the δ -function property $\delta(f(\tilde{t}_r)) = \frac{\delta(t_r - \tilde{t}_r)}{\left| \frac{\partial f(\tilde{t}_r)}{\partial \tilde{t}_r} \right|_{\tilde{t}_r=t_r}}$.

Putting all together:

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\vec{r}_r \frac{\delta(\vec{r}_r - \vec{r}_r)}{|\vec{r} - \vec{w}(\vec{r}_r)|} \frac{1}{1 - \frac{1}{c} \hat{r} \cdot \vec{v}(\vec{r}_r)}$$

\uparrow
 $\frac{\vec{r} - \vec{w}(\vec{r}_r)}{|\vec{r} - \vec{w}(\vec{r}_r)|}$

Using the δ -function is trivial:

$$\varphi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \underbrace{\vec{w}(\vec{r}_r)}_{\vec{r}(t_r)}|} \frac{1}{\left(1 - \frac{1}{c} \hat{r} \cdot \vec{v}(t_r)\right)}$$

$$\text{where } t_r = t - \frac{1}{c} |\vec{r} - \vec{w}(\vec{r}_r)|$$

This is the formula we wanted to prove!