Problem 1, March 2015

(a) The magnetic scalar potential of a magnetic dipole at the origin of coordinates pointing along the z axis is:

$$\Phi_M^{dipole} = \frac{mz}{4\pi r^3} \quad \left(\text{in general it is } \frac{\vec{m}\cdot\vec{x}}{4\pi r^3} \right)$$

In our problem $\vec{J}=0$, thus I can use $\vec{H} = -\nabla\Phi_M$ to solve the problem. Also since $\nabla\cdot\vec{H}=0$ (because $\nabla\cdot\vec{B}=0$ and $\vec{B}=\mu\vec{H}$) then $\nabla^2\Phi_M=0$ is the equation I must solve everywhere.

<u>Outside</u> I propose:

$$\Phi_{M}\Big|_{r>b} = \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

expansion valid for problems with azymuthal symmetry.

In the <u>inner region</u>:

$$\Phi_{M}\Big|_{a<r<b} = \sum_{l=0}^{\infty} \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Inside:

$$\underset{r<a}{\Phi_M} = \underbrace{\frac{mz}{4\pi r^3}} + \sum_{l=0}^{\infty} \delta_l r^l P_l(\cos\theta)$$

$$\frac{m\, r\cos\theta}{4\pi r^3} = \frac{m\cos\theta}{4\pi r^2} = \frac{m}{4\pi r^2} P_1(\cos\theta).$$

Since there is only a "source" for $l=1$, i.e. the center magnetic dipole, then all other l's do not contribute. I can focus only on $l=1$.

The two boundary conditions are:

$$(\vec{B}_2 - \vec{B}_1)\cdot\vec{n} = 0$$

and

$$(\vec{H}_2 - \vec{H}_1)\times\vec{n} = 0$$

For the first equation, $\vec{B} = \mu\vec{H} = -\mu\nabla\Phi_M$. Since $\vec{n} = \hat{e}_r$, then of ∇, I only focus on the $\hat{e}_r$ component which is $\frac{d}{dr}\hat{e}_r$.

$$\vec{B}\cdot\vec{n} \overset{\text{sphere}}{=} -\mu \frac{d}{dr}\Phi_M$$

Now specializing for our two spherical surfaces we get:

$$\boxed{\begin{aligned} &\left(-\mu_0 \frac{d}{dr}\Phi_M^{r>b}\right)\Bigg|_{r=b} - \left(-\mu \frac{d}{dr}\Phi_M^{a<r<b}\right)\Bigg|_{r=b} = 0 \\ &\left(-\mu \frac{d}{dr}\Phi_M^{a<r<b}\right)\Bigg|_{r=a} - \left(-\mu_0 \frac{d}{dr}\Phi_M^{r<a}\right)\Bigg|_{r=a} = 0 \end{aligned}} \quad (\mathrm{II})$$

$$\frac{d\Phi_M^{r>b}}{dr} = \sum_\ell \alpha_\ell \frac{(-\ell-1)}{r^{\ell+2}} P_\ell(\cos\theta) \overset{\ell=1}{=} -\frac{2\alpha_1}{r^3} P_1(\cos\theta)$$

$$\frac{d\Phi_M^{a<r<b}}{dr} = \sum_\ell \left(\beta_\ell \ell r^{\ell-1} + \frac{\gamma_\ell(-\ell-1)}{r^{\ell+2}}\right) P_\ell(\cos\theta) \overset{\ell=1}{=}$$

$$= \left(\beta_1 - \frac{2\gamma_1}{r^3}\right) P_1(\cos\theta)$$

$$\frac{d\Phi_M^{r<a}}{dr} = \sum_{\ell} \delta_\ell \, \ell \, r^{\ell-1} P_\ell(\cos\theta) - \frac{2m}{4\pi r^3} P_1(\cos\theta)$$

$$\overset{\ell=1}{=} \left(\delta_1 - \frac{2m}{4\pi r^3}\right) P_1(\cos\theta).$$

Now going to be boundary conditions (I) we get (after dropping $P_1(\cos\theta)$)

$$\boxed{\begin{aligned} &-\mu_0\left(-\frac{2\alpha_1}{b^3}\right) + \mu\left(\beta_1 - \frac{2\gamma_1}{b^3}\right) = 0 \qquad \text{(III)} \\ &-\mu\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right) + \mu_0\left(\delta_1 - \frac{2m}{4\pi a^3}\right) = 0 \qquad \text{(IV)} \end{aligned}}$$

Now consider the other boundary condition:

$(\vec{H}_2 - \vec{H}_1) \times \hat{e}_r = 0$; $\Delta\vec{H}$ has only components along the $\hat{e}_r$ and $\hat{e}_\theta$ directions by symmetry.

Then, $\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 1 & 0 & 0 \\ \Delta H_r & \Delta H_\theta & 0 \end{vmatrix} = \hat{e}_\phi \, \Delta H_\theta \overset{\text{B. Condition.}}{=} 0$

Then, $\Delta H_\theta = 0$

Since $\vec{H} = -\nabla \Phi_M$ we need the $\hat{e}_\theta$ component of ∇ which is $\frac{1}{r}\frac{\partial}{\partial \theta}$. then:

$$\left(\vec{H}^{r>b} - \vec{H}^{a<r<b} \right) \times \hat{e}_r = 0 \text{ becomes}$$

at $r = b$

$$-\frac{1}{b}\frac{\partial}{\partial \theta} \Phi_M^{r>b} \Big|_b - \left(-\frac{1}{b}\frac{\partial}{\partial \theta} \Phi_M^{a<r<b} \right) \Big|_b = 0$$

For $r = a$ it becomes:

$$-\frac{1}{a}\frac{\partial}{\partial \theta} \Phi_M^{a<r<b} \Big|_a - \left(-\frac{1}{a}\frac{\partial}{\partial \theta} \Phi_M^{r<a} \right) \Big|_a = 0$$

or

$$\boxed{\begin{aligned} \frac{\partial}{\partial \theta} \Phi_M^{r>b} \Big]_b &= \frac{\partial}{\partial \theta} \Phi_M^{a<r<b} \Big]_b \\ \frac{\partial \Phi_M^{a<r<b}}{\partial \theta} \Big]_a &= \frac{\partial}{\partial \theta} \Phi_M^{r<a} \Big]_a \end{aligned}} \qquad \text{II}$$

Focus on $l=1$: (dropping $\frac{\partial}{\partial\theta}P_1(\cos\theta)$)

$$\boxed{\begin{aligned} &\frac{\alpha_1}{b^2} = \beta_1 b + \frac{\gamma_1}{b^2} \qquad \text{V} \\ &\beta_1 a + \frac{\gamma_1}{a^2} = \delta_1 a + \frac{m}{4\pi a^2} \qquad \text{VI} \end{aligned}}$$

(b) Now we have to solve Eqs. III, IV, V, VI.

Students can use Mathematica or other software to do this. I will follow the difficult road of solving by hand.

We can rewrite the equations as:

$$\boxed{\begin{aligned} &\mu(\beta_1 b^3 - 2\gamma_1) = -\mu_0 2\alpha_1 \qquad \text{III} \\ &\mu(\beta_1 a^3 - 2\gamma_1) = \mu_0\left(\delta_1 a^3 - \frac{2m}{4\pi}\right) \qquad \text{IV} \\ &\alpha_1 = \beta_1 b^3 + \gamma_1 \qquad \text{V} \\ &\beta_1 a^3 + \gamma_1 = \delta_1 a^3 + \frac{m}{4\pi} \qquad \text{VI} \end{aligned}}$$

"$\delta_1 a^3$" appears in 2 equations:

$$\delta_1 a^3 = \underbrace{\beta_1 a^3 + \gamma_1 - \frac{m}{4\pi} = \frac{\mu}{\mu_0}(\beta_1 a^3 - 2\gamma_1) + \frac{2m}{4\pi}}$$

or $$\boxed{\gamma_1\left(1 + 2\frac{\mu}{\mu_0}\right) = \beta_1 a^3\left(\frac{\mu}{\mu_0} - 1\right) + \frac{3m}{4\pi}}$$ VII

Also "α_1" appears only in 2 equations:

$$\alpha_1 = \underbrace{\boxed{-\frac{1}{2}\frac{\mu}{\mu_0}(\beta_1 b^3 - 2\gamma_1) = \beta_1 b^3 + \gamma_1}}$$

$$\boxed{\left(\frac{\mu}{\mu_0} - 1\right)\gamma_1 = \beta_1 b^3\left(1 + \frac{\mu}{2\mu_0}\right)}$$ VIII

Define $\frac{\mu}{\mu_0} = \mu' > 1$

$$\gamma_1 = \frac{1}{1+2\mu'}\left[\frac{(\mu'-1)\gamma_1}{b^3(1+\frac{1}{2}\mu')} a^3(\mu'-1) + \frac{3m}{4\pi}\right]$$

$$\gamma_1 = \frac{\frac{1}{1+2\mu'} \cdot \frac{3m}{4\pi}}{1 - \frac{(\mu'-1)^2 a^3}{(1+2\mu')(1+\frac{\mu'}{2})b^3}} = \boxed{\frac{\frac{3m}{4\pi}\left(1+\frac{\mu'}{2}\right)b^3}{(1+2\mu')(1+\frac{\mu'}{2})b^3 - (\mu'-1)^2 a^3}}$$