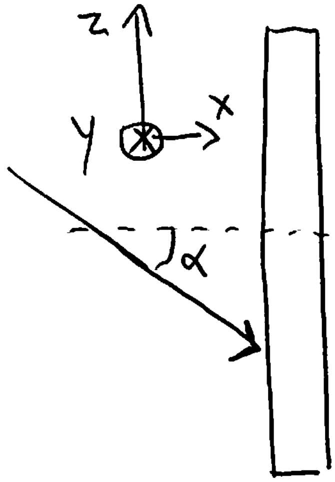
Problem 2
March 2015



$$\vec{E} = E_0 \cos(\omega t)\,\hat{e}_y$$

$$c\,\vec{B}_0 = \vec{n} \times \vec{E}_0 = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \cos\alpha & 0 & -\sin\alpha \\ 0 & E_0 & 0 \end{vmatrix} = (\sin\alpha\, E_0, 0, \cos\alpha\, E_0)$$

$$\vec{B} = \frac{E_0}{c} \cos(\omega t)(\sin\alpha\, \hat{e}_x + \cos\alpha\, \hat{e}_z)$$

(a)

To construct the Maxwell stress tensor note that:

$$E_2 \neq 0, \quad E_1 = E_3 = 0$$

$$B_1 \neq 0, \quad B_3 \neq 0, \; B_2 = 0$$

$$\vec{E}^2 + c^2 \vec{B}^2 = E_0^2 \cos^2(\omega t) + \cancel{c^2} \frac{E_0^2}{\cancel{c^2}} \cos^2(\omega t)\underbrace{(\sin^2\alpha + \cos^2\alpha)}_{=1}$$

$$= 2 E_0^2 \cos^2(\omega t)$$

$$T_{11} = \epsilon_0 \Big[\underbrace{E_1^2}_{=0} + c^2 \underbrace{B_1^2}_{\frac{E_0^2}{c^2}\cos^2(\omega t)\sin^2\alpha} - \frac{1}{2}\, 2 E_0^2 \cos^2(\omega t)\Big]$$

$$= \epsilon_0 E_0^2 \cos^2(\omega t)\,(\sin^2\alpha - 1)$$

$$T_{22} = \epsilon_0 \Big[\underbrace{E_2^2}_{=E_0^2\cos^2(\omega t)} + c^2 \underbrace{B_2^2}_{=0} - \frac{1}{\cancel{2}} \cancel{2} E_0^2 \cos^2(\omega t)\Big] = 0$$

$$T_{33} = \epsilon_0 \Big[\underbrace{E_3^2}_{=0} + c^2 \underbrace{B_3^2}_{=\frac{E_0^2}{c^2}\cos^2(\omega t)\cos^2\alpha} - \frac{1}{2} 2 E_0^2 \cos^2(\omega t)\Big]$$

$$= \epsilon_0 E_0^2 \cos^2(\omega t)(\cos^2\alpha - 1)$$

Now consider the nondiagonal components:

$$T_{\alpha\beta} = \epsilon_0 \left(E_\alpha E_\beta + c^2 B_\alpha B_\beta\right) \quad (\alpha \neq \beta)$$

$$T_{12} = \epsilon_0 \big(\underbrace{E_1}_{=0} E_2 + c^2 B_1 \underbrace{B_2}_{=0}\big) = 0$$

$$T_{13} = \epsilon_0 \big(\underbrace{E_1}_{=0} E_3 + c^2 B_1 B_3\big) = \epsilon_0 \cancel{c^2} \frac{E_0^2}{\cancel{c^2}} \cos^2(\omega t) \sin\alpha\cos\alpha$$

$$T_{23} = \epsilon_0 \big(E_2 \underbrace{E_3}_{=0} + c^2 \underbrace{B_2}_{=0} B_3\big) = 0$$

Then, the 3x3 stress tensor is:

$$\boxed{\hat{T} = \epsilon_0 E_0^2 \cos^2(\omega t) \begin{pmatrix} \sin^2\alpha - 1 & 0 & \sin\alpha\cos\alpha \\ 0 & 0 & 0 \\ \sin\alpha\cos\alpha & 0 & \cos^2\alpha - 1 \end{pmatrix}}$$

If $\alpha = 0$, only T_{11} is non zero as in the problem solved in class.

(b) Now we need the combination $\sum_\beta T_{\alpha\beta} n_\beta$ with $n_\beta = (-1, 0, 0)$

$\underline{\alpha=1}$ $$T_{11} \underset{\substack{\| \\ -1}}{n_1} + T_{12} \underset{\substack{\| \\ 0}}{n_2} + T_{13} \underset{\substack{\| \\ 0}}{n_3} = -T_{11} =$$
$$= -\epsilon_0 E_0^2 \cos^2(\omega t)(\sin^2\alpha - 1)$$

$\underline{\alpha=2}$ $$\underset{\substack{\| \\ 0}}{T_{21}} n_1 + T_{22} \underset{\substack{\| \\ 0}}{n_2} + T_{23} \underset{\substack{\| \\ 0}}{n_3} = 0$$

$\underline{\alpha=3}$ $$T_{31} \underset{\substack{\| \\ -1}}{n_1} + T_{32} \underset{\substack{\| \\ 0}}{n_2} + T_{33} \underset{\substack{\| \\ 0}}{n_3} = -T_{31}$$
$$= -\epsilon_0 E_0^2 \cos^2(\omega t) \sin\alpha\cos\alpha$$

$$\text{Force on screen}\Big|_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, dS =$$

$$= \epsilon_0 E_0^2 \cos^2(\omega t)\, \text{Area} \begin{array}{c} \begin{pmatrix} 1-\sin^2\alpha \\ 0 \\ -\sin\alpha\cos\alpha \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \alpha=1 \\ \leftarrow \alpha=2 \\ \leftarrow \alpha=3 \end{array}$$

The time average of $\cos^2(\omega t)$ is $1/2$. Then:

$$\boxed{\begin{aligned} \text{Force}\Big]_1 &= \frac{\epsilon_0 E_0^2}{2}\left(1-\sin^2\alpha\right) \text{Area} \\ \text{Force}\Big]_3 &= \frac{\epsilon_0 E_0^2}{2}\left(-\sin\alpha\cos\alpha\right) \text{Area} \end{aligned}}$$

If $\alpha=0$, $\text{Force}\Big]_1 = \frac{\epsilon_0 E_0^2}{2}$ as found in class.