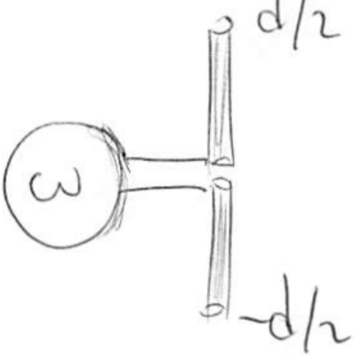
(a, b) **Problem 1**

We follow the same steps as in Section 9.4 (or lecture):

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'$$

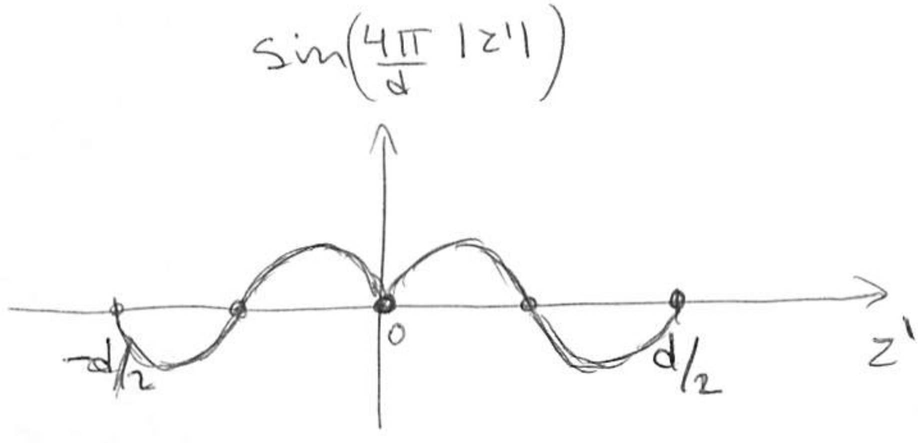
$$\vec{J}(\vec{x}') = I\sin(k|z'|)\,\delta(x')\delta(y')\,\hat{e}_z$$

$(|z'| \leq d/2)$ $(kd = 4\pi$ assumed$)$




Because $r \gg \lambda$ and $d$, we can simplify $\vec{A}$ as in Section 9.4:

$$\boxed{\vec{A}(\vec{x}) \simeq \hat{e}_z\,\frac{I\mu_0}{4\pi}\,\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz'\,\sin(k|z'|)\,e^{-ikz'\cos\theta} \quad \text{with } k = \frac{4\pi}{d}}$$



(c)

The integral we need is:

$$I = \int_{-d/2}^{d/2} dz' \sin(k|z'|)\, e^{-ikz'\cos\theta}$$

for the case $kd = 4\pi$

$$I = I_1 + I_2$$

(1)

$$I_1 = \int_0^{d/2} dz' \sin\left(\frac{4\pi}{d} z'\right) e^{-i\overset{\frac{4\pi}{d}}{k} z'\cos\theta} \overset{\frac{4\pi z'}{d} = u}{=}$$

$$= \int_0^{2\pi} \left(\frac{d}{4\pi}\right) du \,\sin(u)\, e^{-iu\cos\theta} =$$

$$= \left(\frac{d}{4\pi}\right) \int_0^{2\pi} du \left(\frac{e^{iu} - e^{-iu}}{2i}\right) e^{-iu\cos\theta} =$$

$$= \frac{d}{8\pi i} \left[ \int_0^{2\pi} du\, e^{iu(1-\cos\theta)} - \int_0^{2\pi} du\, e^{-iu(1+\cos\theta)} \right] =$$

$$= \frac{d}{8\pi i} \left[ \left. \frac{e^{iu(1-\cos\theta)}}{i(1-\cos\theta)} \right|_0^{2\pi} - \left. \frac{e^{-iu(1+\cos\theta)}}{(-i(1+\cos\theta))} \right|_0^{2\pi} \right] =$$

$$= -\frac{d}{8\pi}\left[\frac{\left(e^{i2\pi(1-\cos\theta)}-1\right)}{(1-\cos\theta)}+\frac{\left(e^{-i2\pi(1+\cos\theta)}-1\right)}{(1+\cos\theta)}\right]$$

$$= -\frac{d}{8\pi}\left[\frac{(1+\cos\theta)\left(e^{i2\pi(1-\cos\theta)}-1\right)+(1-\cos\theta)\left(e^{-i2\pi(1+\cos\theta)}-1\right)}{(1-\cos^2\theta)}\right]$$

$$= \frac{-d}{8\pi}\cdot\left[\frac{(1+\cos\theta)\left(e^{-i2\pi\cos\theta}-1\right)+(1-\cos\theta)\left(e^{-i2\pi\cos\theta}-1\right)}{\sin^2\theta}\right]=$$

$$e^{\pm i2\pi}=1$$

$$=\left(\frac{-d}{8\pi}\right)\frac{\left[\overbrace{(1+\cos\theta+1-\cos\theta)}^{2}\left(e^{-i2\pi\cos\theta}-1\right)\right]}{\sin^2\theta}=$$

$$=\boxed{\frac{d}{4\pi}\cdot\frac{\left(1-e^{-i2\pi\cos\theta}\right)}{\sin^2\theta}=I_1}$$

(2)

$$I_2 = \int_{-d/2}^{0} dz' \, \sin\!\left(\frac{4\pi(-z')}{d}\right) e^{-i k z' \cos\theta}$$

$k = 4\pi/d$

$$-\frac{4\pi z'}{d} = u$$

$$= \int_{2\pi}^{0} \left(-\frac{d}{4\pi}\right) du \, \sin(u) \, e^{-i(-u)\cos\theta} =$$

$$= -\frac{d}{4\pi} \int_{2\pi}^{0} du \, \sin(u) \, e^{iu\cos\theta}$$

$$= \frac{d}{4\pi} \int_{0}^{2\pi} du \, \sin(u) \, e^{iu\cos\theta}$$

same as (1) but replacing $\cos\theta \longrightarrow -\cos\theta$

$$\boxed{I_2 = \frac{d}{4\pi} \cdot \frac{\left(1 - e^{+i 2\pi\cos\theta}\right)}{\sin^2\theta}}$$

$$I_1 + I_2 = \frac{d}{4\pi \sin^2\theta} \cdot \left[2 - \underbrace{\left(e^{i2\pi\cos\theta} + e^{-i2\pi\cos\theta}\right)}_{2\cos(2\pi\cos\theta)}\right]$$

$$= \boxed{\frac{d}{2\pi \sin^2\theta} \cdot \left[1 - \cos(2\pi\cos\theta)\right] = I}$$

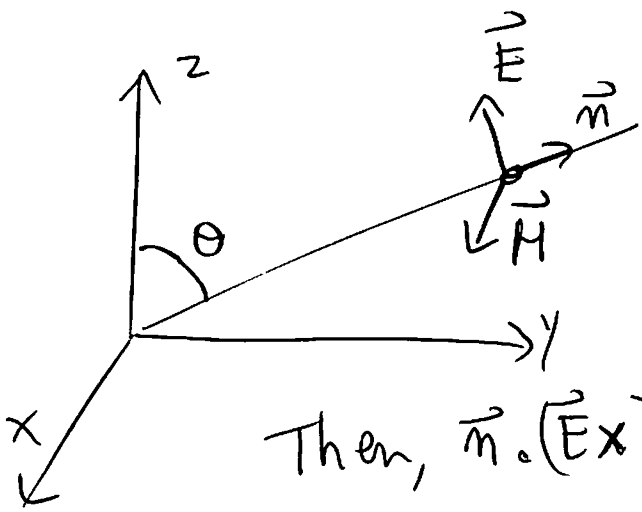
$$\boxed{\vec{A}(\vec{x}) = \hat{e}_z \frac{\mu_0}{4\pi} I \frac{e^{ikr}}{r} \frac{d}{2\pi \sin^2\theta}\left[1 - \cos(2\pi\cos\theta)\right]}$$

Use the generic formula (9.21) for $\frac{dP}{d\Omega}$:
(time averaged)

(d)

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{2} \mathrm{Re}\left[r^2 (\vec{n} \cdot \vec{E} \times \vec{H}^*)\right]}$$

In the radiation zone we know that:




Then, $\vec{n} \cdot (\vec{E} \times \vec{H}^*) = \vec{n} \cdot |\vec{E}||\vec{H}^*|\vec{n}$
$= |\vec{E}||\vec{H}^*|$

We know from plane waves that $c|\vec{B}| = |\vec{E}|$ in the radiation limit, and $|\vec{B}| = \mu_0 |\vec{H}|$ i.e.

$$c\mu_0 |\vec{H}| = |\vec{E}|$$

$$\frac{dP}{d\Omega} = \frac{1}{2} r^2 c^{\cancel{2}} \mu_0^{\cancel{2}} |\vec{H}|^2$$

and using $\underbrace{\boxed{\vec{H} = ik\, \vec{n} \times \frac{\vec{A}}{\mu_0}}}_{\text{given in text}}$ or $|\vec{H}| = \frac{k \sin\theta\, |A_3|}{\mu_0}$

we put:

$$\frac{dP}{d\Omega} = \frac{1}{2} r^2 \underbrace{c\mu_0}_{\frac{1}{\sqrt{\epsilon_0\mu_0}}\mu_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0} k^2 \sin^2\theta \frac{|A_3|^2}{\mu_0^2} =$$

plugging $|A_3|^2$

$$= \frac{1}{2} r^2 Z_0 \frac{k^2}{\mu_0^2} \sin^2\theta \, |A_3|^2 =$$

$$= \frac{1}{2} \cancel{r^2} Z_0 \frac{k^2}{\cancel{\mu_0^2}} \cancel{\sin^2\theta} \frac{\cancel{\mu_0^2}}{(4\pi)^2} I^2 \frac{1}{\cancel{r^2}} \frac{d^2}{(2\pi)^2 \sin^{\cancel{4}2}\theta} \left[1 - \cos(2\pi\cos\theta)\right]^2$$

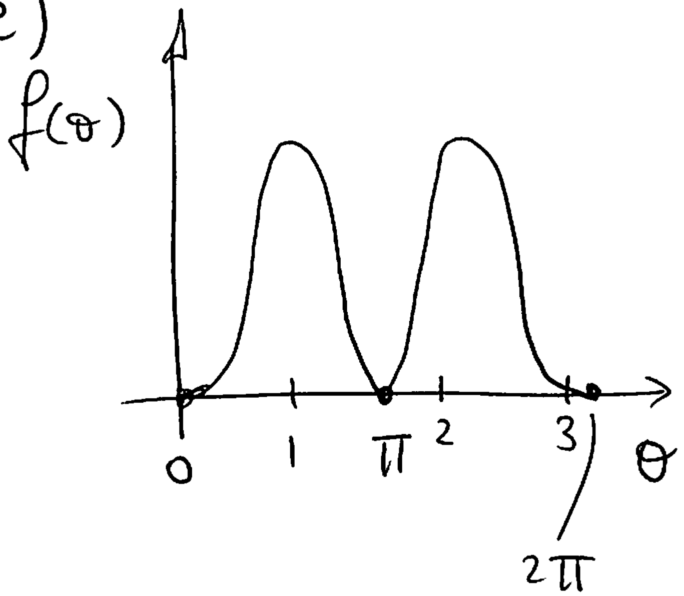
$$= \frac{1}{2} \frac{Z_0 k^2}{(4\pi)^2} I^2 \frac{d^2}{(2\pi)^2 \sin^2\theta} \left(1 - \cos(2\pi\cos\theta)\right)^2$$

$$= \frac{1}{2} \frac{Z_0 k^2 I^2 d^2}{\cancel{(4\pi)^2}(2\pi)^2 \sin^2\theta} \left(1 - \cos(2\pi\cos\theta)\right)^2$$

$k^2 d^2 = (4\pi)^2$

$$= \boxed{\frac{Z_0 I^2}{8\pi^2} \cdot \frac{\left[1 - \cos(2\pi\cos\theta)\right]^2}{\sin^2\theta} = \frac{dP}{d\Omega}}$$

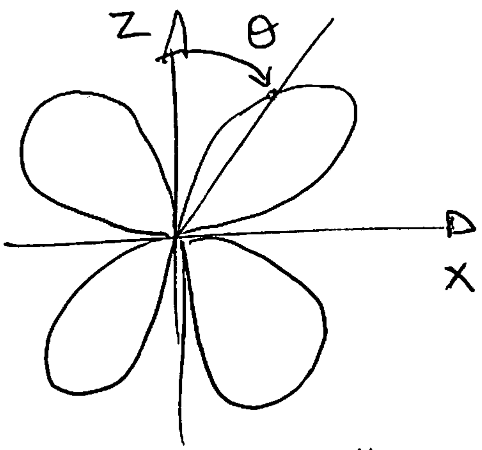
(e)




Plotting the result with a computer

$$f(\theta) = \frac{1 - \cos(2\pi\cos\theta)}{\sin\theta}$$

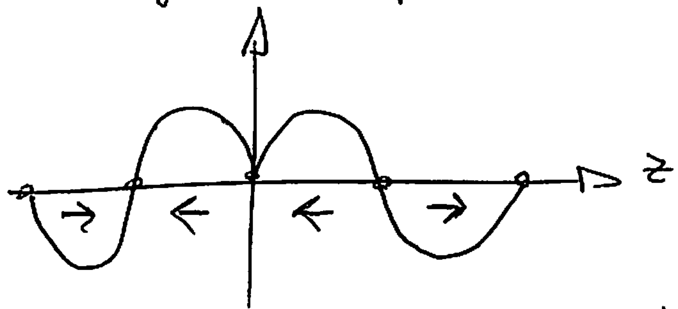
Then, the distribution of radiation is:




(rotational invariance around z axis)

This is a "quadrupole radiation" pattern as in Fig. 9.2 of Jackson

Intuitively, the plot of the current




indicates that this is a combination of "4 dipoles" as indicated by arrows. The total dipole cancels and only the quadrupolar component survives.

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