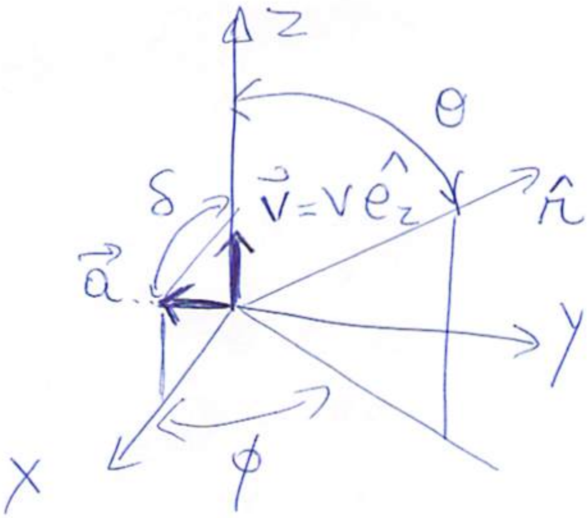
Problem 2



$$\vec{a} = a\left[\cos\delta\,\hat{e}_z + \sin\delta\,\hat{e}_x\right]$$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0}\frac{|\hat{r}\times(\vec{u}\times\vec{a})|^2}{(\hat{r}\cdot\vec{u})^5} \qquad \left(\begin{matrix}11.72\\ \text{Griffiths}\end{matrix}\right)$$

The denominator is the same that appeared in one of the homework problems:

$$\hat{r}\cdot\vec{u} = \hat{r}\cdot(c\hat{r}-\vec{v}) = c - v\underbrace{(\hat{r}\cdot\hat{e}_z)}_{\cos\theta} = c\left(1-\frac{v}{c}\cos\theta\right)$$

For the numerator we use:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\hat{r}\times(\vec{u}\times\vec{a}) = \vec{u}(\hat{r}\cdot\vec{a}) - \vec{a}(\hat{r}\cdot\vec{u})$$

$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = \left[\vec{u}(\hat{r}\cdot\vec{a}) - \vec{a}(\hat{r}\cdot\vec{u})\right]\cdot\left[\vec{u}(\hat{r}\cdot\vec{a}) - \vec{a}(\hat{r}\cdot\vec{u})\right]$$

$$= \underbrace{(\vec{u}\cdot\vec{u})}\,(\hat{r}\cdot\vec{a})^2 - 2\,(\vec{a}\cdot\vec{u})\,(\hat{r}\cdot\vec{a})\underbrace{(\hat{r}\cdot\vec{u})}_{\downarrow\; c\left(1-\frac{v}{c}\cos\theta\right)} + \underbrace{(\vec{a}\cdot\vec{a})}_{a^2}\,\underbrace{(\hat{r}\cdot\vec{u})^2}_{c^2 \times \left(1-\frac{v}{c}\cos\theta\right)^2}$$

$$(c\hat{r}-v\hat{e}_z)^2 =$$

$$= (c\hat{r}-v\hat{e}_z)\cdot(c\hat{r}-v\hat{e}_z)$$

$$= c^2+v^2-2vc\underbrace{(\hat{r}\cdot\hat{e}_z)}_{\cos\theta}$$

$$= \left(c^2+v^2-2vc\cos\theta\right)(\hat{r}\cdot\vec{a})^2 - 2c\left(1-\frac{v}{c}\cos\theta\right)(\vec{a}\cdot\vec{u})\,(\hat{r}\cdot\vec{a})$$

$$+ a^2c^2\left(1-\frac{v}{c}\cos\theta\right)^2$$

$$\hat{r}\cdot\vec{a} = \left(\sin\theta\,\cos\phi\,\hat{e}_x + \sin\theta\,\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z\right)\cdot$$

$$\cdot\left(a\cos\delta\,\hat{e}_z + a\sin\delta\,\hat{e}_x\right)$$

$$= a\left(\sin\theta\cos\phi\sin\delta + \cos\theta\cos\delta\right)$$

$$\vec{a}\cdot\vec{u} = \vec{a}\cdot(c\hat{r}-\vec{v}) = c\underbrace{(\vec{a}\cdot\hat{r})}_{\text{see right above}} - \underbrace{(\vec{a}\cdot\vec{v})}_{\hookrightarrow\, a\cos\delta\, v}$$

$$|\hat{r}\times(\vec{u}\times\vec{a})|^2 = (c^2+v^2-2vc\cos\theta)\,a^2(\sin\theta\cos\phi\sin\delta + \cos\theta\cos\delta)^2$$

$$-2c\left(1-\frac{v}{c}\cos\theta\right)\overbrace{a(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)}^{\hat{r}\cdot\vec{a}}\times$$

$$\times\underbrace{\left[ca(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta) - a\cos\delta\, v\right]}_{(\vec{a}\cdot\vec{u})}$$

$$+a^2c^2\left(1-\frac{v}{c}\cos\theta\right)^2 =$$

$$= a^2c^2\left[\left(1+\frac{v^2}{c^2}-\frac{2v}{c}\cos\theta\right)(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)^2\right.$$

$$-2\left(1-\frac{v}{c}\cos\theta\right)(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)\times$$

$$\times\left[(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)-\frac{v}{c}\cos\delta\right]$$

$$\left.+\left(1-\frac{v}{c}\cos\theta\right)^2\right] =$$

$$= a^2c^2\left[\left(1+\frac{v^2}{c^2}\cancel{-\frac{2v}{c}\cos\theta}-2+\cancel{\frac{2v}{c}\cos\theta}\right)(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)^2\right.$$

$$\left.+\underbrace{2\left(1-\frac{v}{c}\cos\theta\right)\frac{v}{c}\cos\delta}_{\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta}+\left(1-\frac{v}{c}\cos\theta\right)^2\right]$$

$$= a^2c^2\Big[(-1+\beta^2)\,(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)^2$$
$$+ 2(1-\beta\cos\theta)\underbrace{\beta\cos\delta}_{(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)} + (1-\beta\cos\theta)^2\Big]$$

$$= \boxed{a^2c^2\Big[(1-\beta\cos\theta)^2 + 2(1-\beta\cos\theta)\beta\cos\delta\,(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta) - (1-\beta^2)\,(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)^2\Big]}$$

$$\frac{dP}{d\Omega} = \frac{q^2\,a^2c^2}{16\pi^2\epsilon_0\,c^5(1-\beta\cos\theta)^5} \times \Big[(1-\beta\cos\theta)^2 + 2(1-\beta\cos\theta)\overbrace{\beta\cos\delta}^{(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)}$$
$$-(1-\beta^2)(\sin\theta\cos\phi\sin\delta+\cos\theta\cos\delta)^2\Big]$$

$$\frac{1}{\epsilon_0}\frac{1}{c^3} = \frac{1}{\epsilon_0}\frac{1}{c^2}\frac{1}{c} =$$

$$= \frac{1}{\epsilon_0}\epsilon_0\mu_0\frac{1}{c} = \frac{\mu_0}{c}$$

Defining $f(\theta,\phi,\delta) \equiv \sin\theta\cos\phi\sin\delta + \cos\theta\cos\delta$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2a^2\mu_0}{16\pi^2 c}\,\frac{\Big[(1-\beta\cos\theta)^2 + 2\beta\cos\delta\,(1-\beta\cos\theta)\,f(\theta,\phi,\delta) - (1-\beta^2)\,f(\theta,\phi,\delta)^2\Big]}{(1-\beta\cos\theta)^5}}$$

Special cases

① $\vec{v} \perp \vec{a}$ ($\vec{a}$ along x axis, i.e. $\delta = \pi/2$).
$\sin\delta = 1$, $\cos\delta = 0$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2 \mu_0}{16\pi^2 c} \left[\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}\right]}$$

This is the result in Griffith's book, problem 11.16.

② If $\underset{(\beta=0)}{v=0}$, and $\vec{a} = a\hat{e}_z$ (i.e. $\delta = 0$)

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2 \mu_0}{16\pi^2 c} \underbrace{(1 + \cancel{2}\cdot 0 - \cos^2\theta)}_{\sin^2\theta}}$$

which is the expected result from the Larmor formula analysis.

③ $\vec{v} \parallel \vec{a}$ (both along z axis)

$\delta = 0$, $\sin\delta = 0$, $\cos\delta = 1$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2 \mu_0}{16\pi^2 c\,(1-\beta\cos\theta)^5}\left[(1-\beta\cos\theta)^2 + 2\beta(1-\beta\cos\theta)\cos\theta - (1-\beta^2)\cos^2\theta\right]$$

$$1 - \cancel{2\beta\cos\theta} + \cancel{\beta^2\cos^2\theta} + \cancel{2\beta\cos\theta} - \cancel{2\beta^2\cos^2\theta} - \cos^2\theta + \cancel{\beta^2\cos^2\theta}$$

$$= \sin^2\theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2 \mu_0}{16\pi^2 c} \cdot \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

which is the result in Griffith's book, Eq. (11.74)