Problem 3
March 2015

(a) The four Max. Eqs. are:

$$\nabla\cdot\vec{D}=\rho\ , \quad \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$
$$\nabla\cdot\vec{B}=0\ , \quad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

Since this problem is time independent then $\frac{\partial\vec{D}}{\partial t}=\frac{\partial\vec{B}}{\partial t}=0$. In addition $\vec{E}$ and $\vec{D}$ are 0.

Then, we only have

$$\nabla\cdot\vec{B}=0 \quad \text{and} \quad \nabla\times\vec{H}=\vec{J}$$

plus $\begin{cases}\vec{B}=\mu\vec{H} & \text{because is linear}\\ \vec{B}=\nabla\times\vec{A} & \text{by definition}\end{cases}$

and the London Eq. $\vec{J}=-\dfrac{\vec{A}}{K}$

Then, $\nabla\times\vec{B}=\mu\vec{J}=-\dfrac{\mu}{K}\vec{A}$

$$\underbrace{\nabla\times(\nabla\times\vec{B})}=-\frac{\mu}{K}(\nabla\times\vec{A})=-\frac{\mu}{K}\vec{B}$$

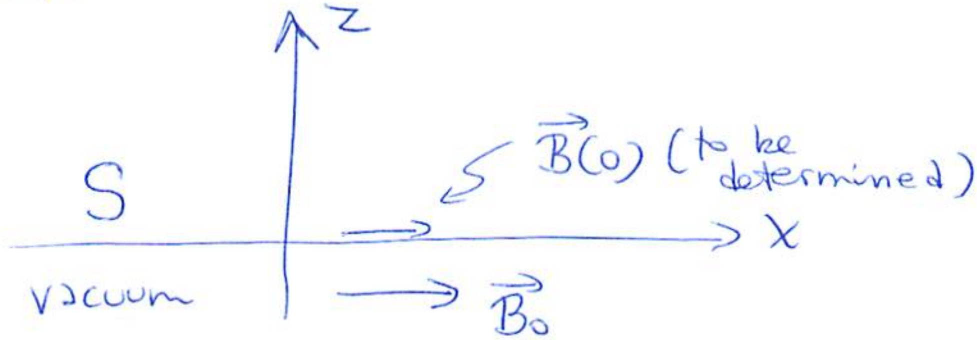
$$\nabla(\underbrace{\nabla\cdot\vec{B}}_{=0})-\nabla^2\vec{B}=-\frac{\mu}{K}\vec{B}$$

$$\boxed{\nabla^2\vec{B}=\frac{\mu}{K}\vec{B}}$$

(b) To solve the equation found in (a) let us try $\vec{B} = \vec{B}(0)e^{-z/\lambda}$

where $\vec{B}(0)$ points along the x axis (see sketch)

i.e. $\vec{B}(0) = B(0)\,\hat{e}_x$



$$\nabla^2\vec{B} = \nabla^2 B(0)e^{-z/\lambda}\,\hat{e}_x = B(0)\underbrace{\left(\frac{d^2}{dz^2}e^{-z/\lambda}\right)}_{\frac{1}{\lambda^2}e^{-z/\lambda}}\hat{e}_x$$

$$= \frac{1}{\lambda^2}\vec{B}$$

Then, $\boxed{\lambda = \sqrt{\frac{K}{\mu}}}$ (London penetration depth)

The exponential decay corresponds to the Meissner effect. It is similar to the exponential suppression we found in class when in the presence of an oscillating field (skin depth effect). But in superconductors the Meissner effect happens even in a <u>time independent</u> context.

We still have to find $B(0)$ and for this purpose we must use the boundary conditions, in particular "$(\vec{H}_2 - \vec{H}_1) \times \vec{n}_{21} = 0$".

$$\vec{n}_{21} = (0, 0, 1)$$

$$\vec{H}_2 - \vec{H}_1 = (\Delta H, 0, 0) \leftarrow \vec{H} \text{ only oriented along } x \text{ axis.}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta H & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\Delta H \, \hat{\jmath} = 0$$

Then $H(\text{at } z = 0 - \varepsilon) = H(\text{at } z = 0 + \varepsilon)$

But $\vec{H} = \dfrac{\vec{B}}{\mu}$: $\quad \dfrac{B(\text{at } z = 0 - \varepsilon)}{\mu_0} = \dfrac{B(\text{at } z = 0 + \varepsilon)}{\mu}$

$$B(\text{at } z = 0 - \varepsilon) = B_0$$
$$B(\text{at } z = 0 + \varepsilon) = B(0)$$

Then $B(0) = \dfrac{\mu}{\mu_0} B_0$

The final answer is :

$$\boxed{\vec{B} = \frac{\mu}{\mu_0} B_0 \, e^{-z/\lambda} \, \hat{e}_x}$$