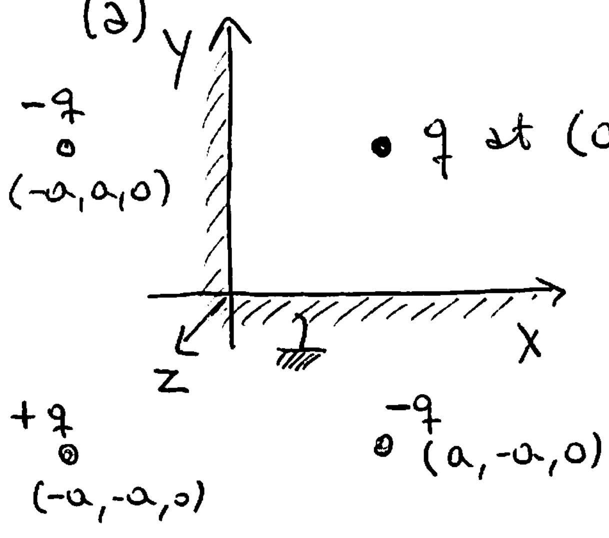
(a)



3 images!

I need an arrangement of charges such that the potential cancels at the surfaces shaded.

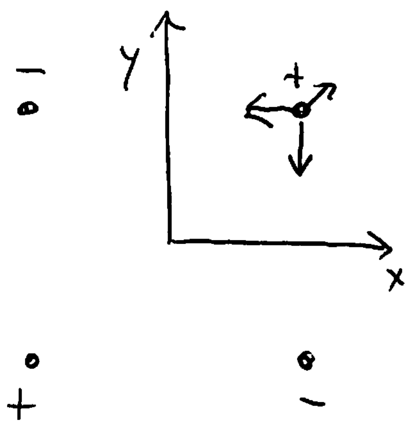
If I use only one image $-q$, say at $(-a,a,0)$ I can cancel the potential at the zy plane but not at the other plane. If I use both $(-a,a,0)$ and $(a,-a,0)$, the contribution of the three does not cancel at each plane.

Then, I need 3 images! Together with the real charge I have a total of 4 and by pairs they produce the desired result. The total potential at an arbitrary point (x,y,z) in the first quadrant is:

$$\boxed{\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-a)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+a)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-a)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+a)^2+z^2}}\right]}$$

It is easy to confirm that this potential cancels at $x=0$ and at $y=0$.

(b) The force at the real charge is made of 3 terms, due to the images.



$$\vec{F}_{\text{at }(a,a,0)} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\hat{e}_x}{(2a)^2} - \frac{\hat{e}_y}{(2a)^2} + \frac{\hat{e}_x+\hat{e}_y}{\sqrt{2}\,(\sqrt{2}\,2a)^2} \right]$$

to normalize unit vector to 1

distance from $(a,a,0)$ to $(-a,-a,0)$

$$= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{\hat{e}_x}{4a^2} - \frac{\hat{e}_y}{4a^2} + \frac{(\hat{e}_x+\hat{e}_y)}{8a^2\sqrt{2}} \right].$$

(c) To find the Dirichlet Green function $G_D(\vec{x},\vec{x}')$ we use the discussion in the book and lectures involving the analogy of G_D with the potential Φ found in (a). Basically we have to locate the charge q at an arbitrary point (x',y',z') instead of $(a,a,0)$, find the images at $(-x',y',z')$, $(x',-y',z')$ and $(-x',-y',z')$ and the get the potential of $q=4\pi\epsilon_0$.

$$G_D(\vec{x},\vec{x}') = \left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x+x')^2+(y+y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y+y')^2+(z-z')^2}}\right]$$

At $y'=0$ it cancels. At $x'=0$ it cancels.

(d) Formally the surface integral that I need is

$$\text{“} -\frac{1}{4\pi}\oint_S \Phi(x')\frac{\partial G_D}{\partial n'}\,da' \text{”} \quad \text{from the book.}$$

In our case the surface is the top half plane at $x'=0$

So "$\oint_S$" means $\int_0^{\infty} dy' \int_{-\infty}^{+\infty} dz'$. "$\Phi(x')$" is just a constant $\mathbf{V}$.

"da'" is just $dy'dz'$, already written in the integral.

"$\frac{\partial G_D}{\partial n'}$" means $\nabla G_D \cdot \vec{n}'$ and $\vec{n}'$ is $(-\hat{e}_x)$. So $\nabla G_D \cdot \vec{n}'$

means $-\frac{d}{dx'} G_D(\vec{x},\vec{x}')$ at $x'=0$.

from the volume considered outwards.

$$
\begin{aligned}
\frac{d}{dx'} G_D(\vec{x},\vec{x}') = & \frac{d}{dx'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2} \\
& + \frac{d}{dx'}\left[(x+x')^2+(y+y')^2+(z-z')^2\right]^{-1/2} \\
& + \frac{d}{dx'}\left[-\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}\right] \\
& + \frac{d}{dx'}\left[-\left[(x-x')^2+(y+y')^2+(z-z')^2\right]^{-1/2}\right] =
\end{aligned}
$$

$$
\begin{aligned}
= & -\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} 2(x-x')(-1) \\
& -\frac{1}{2}\left[(x+x')^2+(y+y')^2+(z-z')^2\right]^{-3/2} 2(x+x') \\
& +\frac{1}{2}\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} 2(x+x') \\
& +\frac{1}{2}\left[(x-x')^2+(y+y')^2+(z-z')^2\right]^{-3/2} 2(x-x')(-1)
\end{aligned}
$$

Now I have to take $x'=0$ to be at the surface.

$$
\begin{aligned}
\left.\frac{d}{dx'} G_D(\vec{x},\vec{x}')\right|_{\substack{\text{surface} \\ x'=0}} = & \; x\left[x^2+(y-y')^2+(z-z')^2\right]^{-3/2} - x\left[x^2+(y+y')^2+(z-z')^2\right]^{-3/2} \\
& + x\left[x^2+(y-y')^2+(z-z')^2\right]^{-3/2} - x\left[x^2+(y+y')^2+(z-z')^2\right]^{-3/2}
\end{aligned}
$$

$$
= \frac{2x}{\left[x^2+(y-y')^2+(z-z')^2\right]^{3/2}} - \frac{2x}{\left[x^2+(y+y')^2+(z-z')^2\right]^{3/2}}.
$$

The final expression then is:

$$\boxed{\Phi(\vec{x}) = + \frac{\mathbf{V}}{4\pi} \int_0^{\infty} dy' \int_{-\infty}^{+\infty} dz' \; 2x \left[\frac{1}{\left[x^2+(y-y')^2+(z-z')^2\right]^{3/2}} - \frac{1}{\left[x^2+(y+y')^2+(z-z')^2\right]^{3/2}} \right]}$$

remember $\vec{n}' = -\hat{e}_x$

$\vec{x} = (x, y, z)$ is a point in the first quadrant

(y', z') are coordinates of a point at the surface at constant potential $\mathbf{V}$.

(e)

We use $\displaystyle\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2(a^2+u^2)^{1/2}}$.

(i)

$$\int_{-\infty}^{+\infty} dz' \frac{1}{\sqrt{x^2+(y-y')^2+(z'-z)^2}} \underset{z'-z=u}{=} \int_{-\infty}^{\infty} \frac{du}{\sqrt{\underbrace{x^2+(y-y')^2}_{a^2}+u^2}} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{a^2+u^2}} =$$

$$= \left.\frac{u}{a^2(a^2+u^2)^{1/2}}\right|_{-\infty}^{\infty} = \lim_{u\to\infty} \frac{u}{a^2(a^2+u^2)^{1/2}} - \lim_{u\to-\infty} \frac{u}{a^2(a^2+u^2)^{1/2}}$$

$$= \frac{1}{a^2} + \frac{1}{a^2} = \frac{2}{a^2} = \frac{2}{x^2+(y-y')^2}.$$

(ii) Same but with "a^2" $= x^2+(y+y')^2$

Then $\displaystyle\int_{-\infty}^{+\infty} dz' \left[\frac{1}{(\quad)^{3/2}} - \frac{1}{(\quad)^{3/2}}\right] = \frac{2}{x^2+(y-y')^2} - \frac{2}{x^2+(y+y')^2}$.

What remains is:

$$\frac{+V}{4\pi} 2x \int_0^{\infty} dy' \, 2 \left[\frac{1}{x^2+(y-y')^2} - \frac{1}{x^2+(y+y')^2}\right] =$$

$$= +\frac{Vx}{\pi}\left[\int_0^\infty dy' \, \frac{1}{x^2+\underbrace{(y-y')^2}_{\text{or } (y'-y)^2}} - \int_0^\infty dy' \, \frac{1}{x^2+(y+y')^2}\right]$$

$$= +\frac{Vx}{\pi}\left[\int_{\underset{y'-y=u}{-y}}^\infty du \, \frac{1}{x^2+u^2} - \int_{\underset{y'+y=u}{+y}}^\infty du \, \frac{1}{x^2+u^2}\right]$$

$$= +\frac{Vx}{\pi}\left[\frac{1}{x}\tan^{-1}\left(\frac{u}{x}\right)\Big|_{-y}^{\infty} - \frac{1}{x}\tan^{-1}\left(\frac{u}{x}\right)\Big|_{+y}^{\infty}\right]$$

$$= +\frac{V}{\pi}\left[\cancel{\tan^{-1}(\infty)} - \tan^{-1}\left(-\frac{y}{x}\right) - \left(\cancel{\tan^{-1}(\infty)} - \tan^{-1}\left(\frac{y}{x}\right)\right)\right]$$

$$= \frac{V}{\pi}\left[\tan^{-1}\left(\frac{y}{x}\right) - \underbrace{\tan^{-1}\left(\frac{-y}{x}\right)}_{\text{odd function}}\right] = \frac{2V}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Phi(\vec{x}) = +\frac{2V}{\pi}\tan^{-1}\left(\frac{y}{x}\right) =$$

indep. of z which is correct by symmetry

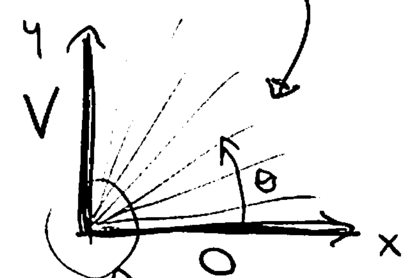
If $y=0$, $\tan^{-1}(0) = 0$ which is the correct boundary condition!

If $x=0$, we get $\Phi(\vec{x}) = +\frac{2V}{\pi}\cdot\frac{\pi}{2}$

$= +V$ which is correct!

$$\boxed{\Phi(\vec{x}) = \frac{2V}{\pi}\tan^{-1}\left(\frac{y}{x}\right) \quad \text{(indep. of } z\text{)}}$$

for any point in.



$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$. So $\boxed{\Phi(\vec{x}) = \frac{2V}{\pi}\theta}$

Huge fields here because $\nabla\phi$ (edge) is huge.