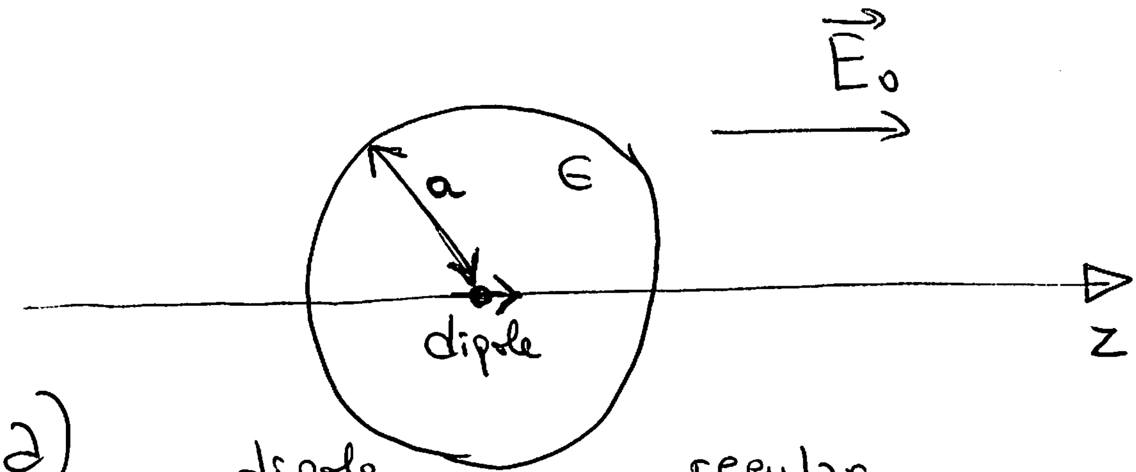

3

(a)

$$\Phi_{in} = \underbrace{\frac{p\cos\theta}{4\pi\epsilon r^2}}_{\text{dipole}} + \underbrace{\sum_l A_l r^l P_l(\cos\theta)}_{\text{regular}}$$

$$\Phi_{out} = -E_0 r\cos\theta + \underbrace{\sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta)}_{\text{regular}}$$

Requesting $\Phi_{in} = \Phi_{out}$ at surface and noting that $\cos\theta = P_1(\cos\theta)$, we get:

$$\frac{p}{4\pi\epsilon a^2} P_1(\cos\theta) + \sum_l A_l a^l P_l(\cos\theta) = -E_0 a P_1(\cos\theta) + \sum_l B_l \frac{1}{a^{l+1}} P_l(\cos\theta)$$

For $l=1$ we get:

$$\frac{p}{4\pi\epsilon a^2} + A_1 a = -E_0 a + \frac{B_1}{a^2}$$

For $l \neq 1$, we get:

$$A_l \, a^l = B_l \frac{1}{a^{l+1}}$$

Another equation we can use is

"$(\vec{D}_2 - \vec{D}_1) \cdot \vec{n}_{21} = 0$" (no free charge)

$$\epsilon \left. \frac{\partial \Phi_{in}}{\partial r} \right|_{at\, r=a} = \epsilon_0 \left. \frac{\partial \Phi_{out}}{\partial r} \right|_{at\, r=a}$$

$$\epsilon \left[\frac{p \cos\theta}{4\pi\epsilon} \frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) + \sum_l A_l P_l(\cos\theta) \frac{\partial}{\partial r}(r^l) \right]_{at\, r=a} =$$

$$= \epsilon_0 \left[-E_0 \cos\theta + \sum_l B_l \frac{\partial}{\partial r}\left(\frac{1}{r^{l+1}}\right) P_l(\cos\theta) \right]_{at\, r=a} ;$$

$$\epsilon \left[\frac{p \overbrace{\cos\theta}^{P_1}}{4\pi\epsilon} \left(\frac{-2}{a^3}\right) + \sum_l A_l P_l(\cos\theta) \, l \, a^{l-1} \right] =$$

$$= \epsilon_0 \left[-E_0 \underbrace{\cos\theta}_{P_1} + \sum_l B_l \frac{(-l-1)}{a^{l+2}} P_l(\cos\theta) \right]$$

For $\ell = 1$:

$$-\frac{2p}{4\pi a^3} + \epsilon A_1 = -\epsilon_0 E_0 - \epsilon_0 \frac{2B_1}{a^3}$$

For $\ell \neq 1$:

$$\epsilon A_\ell \, \ell \, a^{\ell-1} = -\epsilon_0 \frac{(\ell+1) B_\ell}{a^{\ell+2}}$$

Putting all together:

For $\ell \neq 1$, there is no $\underbrace{}_{\text{nonzero}}$ solution i.e. $A_\ell = B_\ell = 0$.

For $\ell = 1$, we have two equations:

$$\frac{p}{4\pi\epsilon a^2} + A_1 a = -E_0 a + \frac{B_1}{a^2} \quad ①$$

and

$$-\frac{2p}{4\pi a^3} + \epsilon A_1 = -\epsilon_0 E_0 - \frac{2\epsilon_0 B_1}{a^3} \quad ②$$

From ①: $A_1 = -E_0 + \frac{B_1}{a^3} - \frac{p}{4\pi\epsilon a^3}$

Replacing in ②:

$$-\frac{2p}{4\pi a^3} + \epsilon\left(-E_0 + \frac{B_1}{a^3} - \frac{p}{4\pi\epsilon a^3}\right) = -\epsilon_0 E_0 - \frac{2\epsilon_0 B_1}{a^3}$$

$$\frac{\epsilon}{a^3} B_1 + \frac{2\epsilon_0}{a^3} B_1 = -\epsilon_0 E_0 + \epsilon E_0 + \frac{2P}{4\pi a^3} + \frac{P}{4\pi a^3}$$

$$B_1 \frac{(\epsilon + 2\epsilon_0)}{a^3} = (\epsilon - \epsilon_0) E_0 + \frac{3P}{4\pi a^3}$$

$$\boxed{B_1 = a^3 \frac{(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} E_0 + \frac{3P}{4\pi (\epsilon + 2\epsilon_0)}}$$

$$A_1 = -E_0 - \frac{P}{4\pi \epsilon a^3} + \frac{1}{a^3}\left[a^3 \frac{(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} E_0 + \frac{3P}{4\pi(\epsilon + 2\epsilon_0)} \right]$$

$$= -E_0 - \frac{P}{4\pi \epsilon a^3} + \frac{(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} E_0 + \frac{3P}{4\pi (\epsilon + 2\epsilon_0) a^3}$$

$$= \left[\frac{-(\epsilon + 2\epsilon_0) + (\epsilon - \epsilon_0)}{\epsilon + 2\epsilon_0} \right] E_0 + \frac{P}{4\pi a^3} \left[-\frac{1}{\epsilon} + \frac{3}{\epsilon + 2\epsilon_0} \right]$$

$$= \frac{-3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 + \frac{P}{4\pi a^3} \frac{(-\epsilon - 2\epsilon_0 + 3\epsilon)}{\epsilon(\epsilon + 2\epsilon_0)}$$

$$= \boxed{\frac{-3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 + \frac{P}{4\pi a^3} \frac{2(\epsilon - \epsilon_0)}{\epsilon(\epsilon + 2\epsilon_0)} = A_1}$$

The final solution then is:

$$\Phi_{in} = \frac{p\cos\theta}{4\pi\epsilon r^2} + \left[\frac{-3\epsilon_0 E_0}{\epsilon+2\epsilon_0} + \frac{p}{4\pi a^3}\left(\frac{2(\epsilon-\epsilon_0)}{\epsilon(\epsilon+2\epsilon_0)}\right)\right]\cos\theta . r$$

$$\Phi_{out} = -E_0 r\cos\theta + \left[\frac{a^3(\epsilon-\epsilon_0)}{(\epsilon+2\epsilon_0)}E_0 + \frac{3p}{4\pi(\epsilon+2\epsilon_0)}\right]\frac{\cos\theta}{r^2}$$

(b)

If $\epsilon = \epsilon_0$ then

$$\Phi_{in} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} - E_0 r\cos\theta$$

$$\Phi_{out} = -E_0 r\cos\theta + \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

Identical as it has to be.

(c) Consider $E_0 = 0$ for simplicity:

$$\Phi_{in} = \frac{p\cos\theta}{4\pi\epsilon r^2} + \frac{p}{4\pi a^3}\frac{2(\epsilon-\epsilon_0)}{\epsilon(\epsilon+2\epsilon_0)} r\cos\theta$$

$$\Phi_{out} = \frac{3p}{4\pi(\epsilon+2\epsilon_0)}\frac{\cos\theta}{r^2}$$

$$\vec{E} = -\nabla \Phi = -\hat{e}_r \frac{\partial \Phi}{\partial r} - \hat{e}_\theta \frac{1}{r} \frac{\partial \Phi}{\partial \theta}$$

$$\boxed{\begin{aligned} \vec{E}_{in} = & -\hat{e}_r \left[\frac{-2p\cos\theta}{4\pi\epsilon r^3} + \frac{p}{4\pi a^3} \frac{2(\epsilon-\epsilon_0)}{\epsilon(\epsilon+2\epsilon_0)} \cos\theta \right] \\ & - \frac{\hat{e}_\theta}{r} \left(\frac{-p\sin\theta}{4\pi\epsilon r^2} - \frac{2p}{4\pi a^3} \frac{(\epsilon-\epsilon_0)\, r \sin\theta}{\epsilon(\epsilon+2\epsilon_0)} \right) \end{aligned}}$$

$$\vec{P}_{in} = (\epsilon-\epsilon_0)\vec{E}_{in} = (\epsilon-\epsilon_0) \Bigg[\underbrace{\frac{2p}{4\pi\epsilon r^3} \cos\theta\, \hat{e}_r + \frac{p\sin\theta}{4\pi\epsilon r^3}\, \hat{e}_\theta}_{\text{contribution of dipole at center}}$$

$$\underbrace{- \frac{2p(\epsilon-\epsilon_0)}{4\pi\epsilon a^3(\epsilon+2\epsilon_0)} \cos\theta\, \hat{e}_r + \frac{2p(\epsilon-\epsilon_0)}{4\pi a^3 \epsilon(\epsilon+2\epsilon_0)} \sin\theta\, \hat{e}_\theta}_{\text{Contribution of polarization of the medium}} \Bigg]$$

$$\frac{2p(\epsilon-\epsilon_0)}{4\pi\epsilon a^3(\epsilon+2\epsilon_0)} \underbrace{\left(-\cos\theta\, \hat{e}_r + \sin\theta\, \hat{e}_\theta \right)}_{-\hat{e}_z}$$

(d) The last term of the previous page, which points in the $(-\hat{e}_z)$ direction and it is constant, is caused by the bound charges.

Consider the fixed dipole at the center and how its presence will polarize the material. Next to the "+" of the fixed dipole we will generate a "−" of the next molecule and so on, as shown in the sketch:

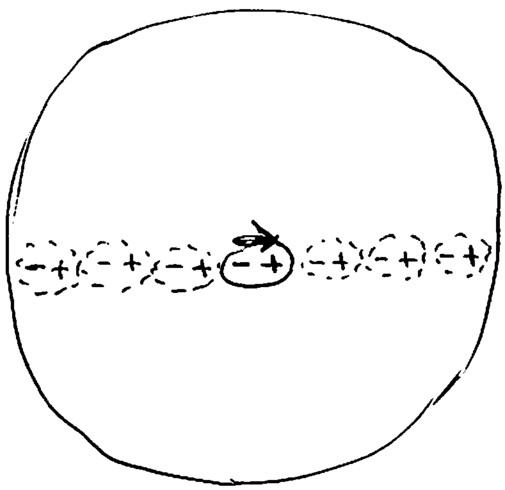

Fig. 1

As discussed in Griffiths, + pluses and − minuses cancel inside but not at the surface leading to Figure 2.

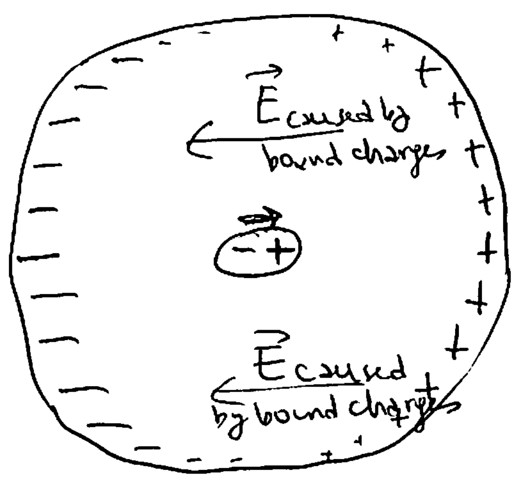

Fig. 2

Then, the electric field caused by the bound charges points in the negative direction of the z-axis, as found mathematically.