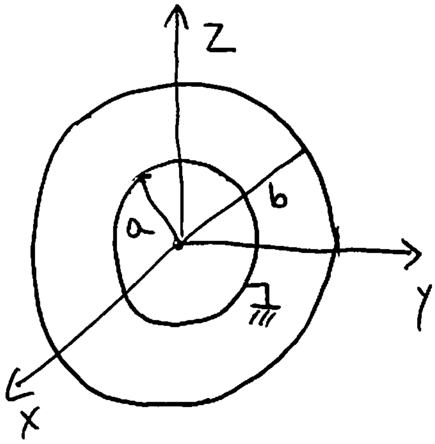


Problem 1

$$\phi\Big|_{\text{at } b} = V_0 \sin\theta \cos\phi$$

(a) From tables: $Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$

$Y_{1-1} = +\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$

Then $Y_{11} - Y_{1-1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \left(e^{i\phi} + e^{-i\phi}\right)$

$$= \underbrace{-\sqrt{\frac{3}{8\pi}}\, 2}_{\gamma} \sin\theta \cos\phi = \gamma \sin\theta \cos\phi$$

$$\boxed{\phi\Big|_{\text{at } b} = \frac{V_0}{\gamma}\left(Y_{11} - Y_{1-1}\right)} = V_0 \sin\theta \cos\phi .$$

(b)

$$\phi\Big|_{a \leq r \leq b} = \sum_{lm} \left(A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}}\right) Y_{lm}(\theta, \phi)$$

Boundary conditions are:

$$\phi\Big|_{\text{at } r=b} = \frac{V_0}{\gamma}(Y_{11} - Y_{1-1}) = \sum_{\ell m}\left(A_{\ell m} b^{\ell} + B_{\ell m}\frac{1}{b^{\ell+1}}\right) Y_{\ell m}(\theta,\phi)$$

$$\phi\Big|_{\text{at } r=a} = 0 = \sum_{\ell m}\left(A_{\ell m} a^{\ell} + B_{\ell m}\frac{1}{a^{\ell+1}}\right) Y_{\ell m}(\theta,\phi)$$

Since the boundary condition only involve Y_{11} and Y_{1-1} it is natural to assume that only $\ell = 1$, $m = \pm 1$ are important.

$$\frac{V_0}{\gamma}(Y_{11} - Y_{1-1}) = \left(A_{11} b + B_{11}\frac{1}{b^2}\right) Y_{11} + \left(A_{1-1} b + B_{1-1}\frac{1}{b^2}\right) Y_{1-1}$$

$$0 = \left(A_{11} a + B_{11}\frac{1}{a^2}\right) Y_{11} + \left(A_{1-1} a + B_{1-1}\frac{1}{a^2}\right) Y_{1-1}$$

Then, the actul eqs are:

$$\underbrace{\begin{aligned} \frac{V_0}{\gamma} &= A_{11} b + B_{11}\frac{1}{b^2} \\ 0 &= A_{11} a + B_{11}\frac{1}{a^2} \end{aligned}}_{\text{System 1}} \quad ; \quad \underbrace{\begin{aligned} -\frac{V_0}{\gamma} &= A_{1-1} b + B_{1-1}\frac{1}{b^2} \\ 0 &= A_{1-1} a + B_{1-1}\frac{1}{a^2} \end{aligned}}_{\text{System 2}}$$

Note that if I solve system 1, then system 2 is obtained by $A_{1-1} = -A_{11}$
$B_{1-1} = -B_{11}$

So it is enough to solve one system.

$$A_{11} = -\frac{B_{11}}{a^3}$$

$$\frac{V_0}{\gamma} = A_{11} b - A_{11} a^3 \frac{1}{b^2} \; ; \; A_{11} = \frac{V_0/\gamma}{b - \frac{a^3}{b^2}} = \frac{V_0}{\gamma} \cdot \frac{b^2}{(b^3-a^3)}$$

$$B_{11} = -\frac{V_0}{\gamma} \frac{a^3 b^2}{(b^3-a^3)}$$

$$A_{1-1} = -A_{11} \quad ; \quad B_{1-1} = -B_{11}$$

$$\phi_{a \le r \le b} = \left(A_{11} r + B_{11} \frac{1}{r^2}\right) Y_{11} + \left(A_{1-1} r + B_{1-1} \frac{1}{r^2}\right) Y_{1-1}$$

$$= A_{11} r Y_{11} - A_{11} r Y_{1-1} + B_{11} \frac{1}{r^2} Y_{11} - B_{11} \frac{1}{r^2} Y_{1-1}$$

$$= \frac{V_0}{\gamma} \frac{b^2}{(b^3-a^3)} r \left(Y_{11} - Y_{1-1}\right) + \left(-\frac{V_0}{\gamma}\right) \frac{a^3 b^2}{(b^3-a^3)} \frac{1}{r^2} \left(Y_{11} - Y_{1-1}\right)$$

$$= \frac{b^2 r}{(b^3-a^3)} V_0 \sin\theta \cos\phi - \frac{a^3 b^2}{(b^3-a^3)} \frac{1}{r^2} V_0 \sin\theta \cos\phi$$

$$= \boxed{\frac{b^2}{(b^3-a^3)} \left(r - \frac{a^3}{r^2}\right) V_0 \sin\theta \cos\phi = \phi_{a \le r \le b}}$$

If $r=a$, $a-\frac{a^3}{a^2}=0$. Correct

If $r=b$, $\frac{b^2}{(b^3-a^3)}\left(b-\frac{a^3}{b^2}\right)=1$. Correct

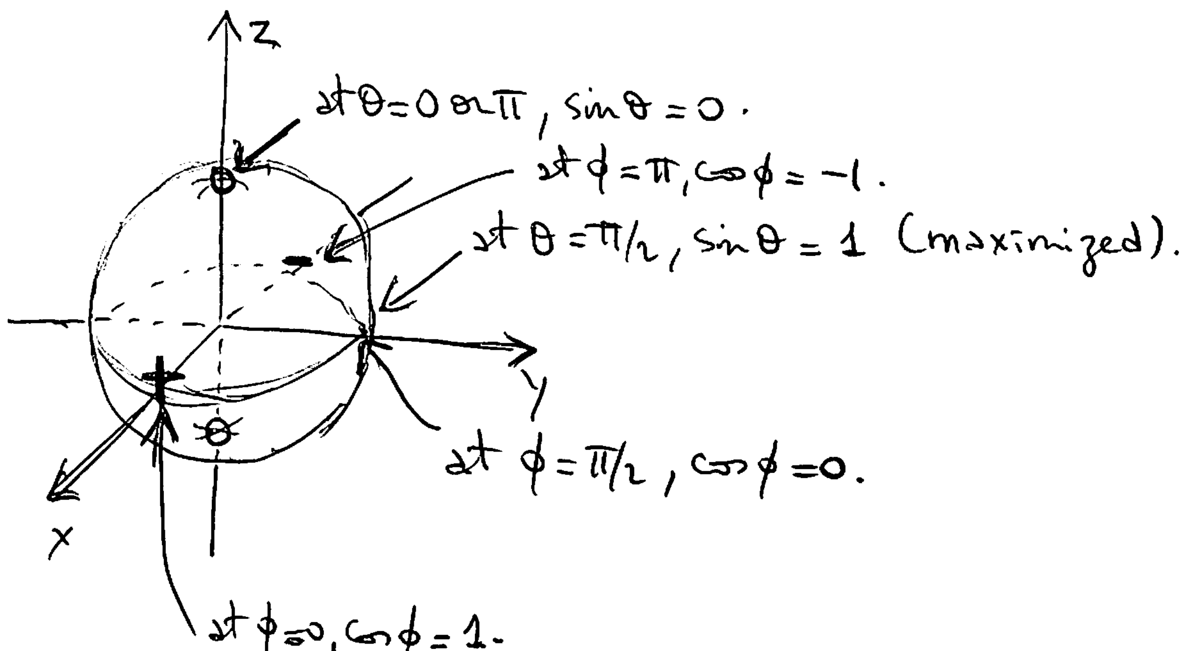
(c) $\sigma=-\epsilon_0 \left.\frac{\partial \phi}{\partial r}\right|_{r=a} = -\frac{\epsilon_0 b^2}{(b^3-a^3)} V_0 \sin\theta\cos\phi \left.\frac{\partial}{\partial r}\left(r-\frac{a^3}{r^2}\right)\right|_{r=a} =$

$$= -\frac{\epsilon_0 b^2}{(b^3-a^3)} V_0 \sin\theta\cos\phi \underbrace{\left.\left(1-\frac{a^3(-2)}{r^3}\right)\right|_{r=a}}_{1+2\frac{a^3}{a^3}=3}$$

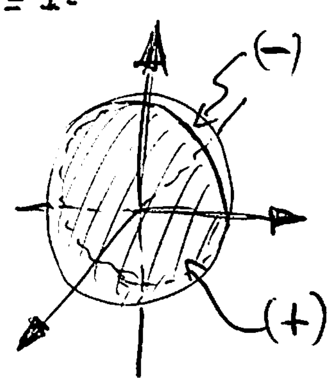
$$\boxed{\sigma=-\frac{\epsilon_0 3b^2}{(b^3-a^3)} V_0 \sin\theta\cos\phi}$$

Note the "−" in front.
σ is "antiphase" with $V_0 \sin\theta\cos\phi$
which is the potential at the surface.

(d)



It is like two hemispheres with different signs, along the x axis.



The internal sphere grounded reacts "antiphase". From the Z axis perspective, which makes sense.