Problem 2

From lecture or from HW3 problem 1 we know that the Dirichlet Green function for a plane is:

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \underbrace{\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}_{\text{Contribution of "image"}}$$

Formula (1.44) says:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x'$$

$$-\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\,\frac{\partial G_D}{\partial n}\,da'$$

Since

$\rho(\vec{x}') = q\,\delta(\vec{x}'-\vec{a})$ in this case, then the <u>first term</u> becomes:

$$\frac{1}{4\pi\epsilon_0}\,q\int_V \delta(\vec{x}'-\vec{a})\,G_D(\vec{x},\vec{x}')\,d^3x' = \frac{q}{4\pi\epsilon_0}\,G_D(\vec{x},\vec{a}) =$$

($\vec{a}$ is $(0,0,a)$)

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right]$$

which physically is the contribution of the real charge plus its image as expected.

With regards to the <u>second term</u>

$$\frac{\partial G_D}{\partial n} = \nabla G_D \cdot \vec{n} \underset{\uparrow}{=} -\frac{\partial G_D}{\partial z'}$$

$\vec{n}$ points from volume i.e. $\vec{n} = (0,0,-1)$

Then the second term becomes

$$\frac{1}{4\pi}\int_{-a/2}^{a/2} dx' \int_{-a/2}^{a/2} dy' \, V \frac{\partial}{\partial z'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]_{\text{at } z'=0}$$

$$\frac{\partial}{\partial z'}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = -\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} \cdot 2(z-z')(-1)$$

$$= \frac{z-z'}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$\frac{\partial}{\partial z'}\left[\frac{-1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right] = \frac{1}{2}\left[(x-x')^2+(y-y')^2+(z+z')^2\right]^{-3/2} \cdot 2(z+z')$$

$$= \frac{z+z'}{\left[(x-x')^2+(y-y')^2+(z+z')^2\right]^{3/2}}$$

The seond term thus becomes

$$\frac{1}{4\pi}\int_{-a/2}^{a/2} dx' \int_{-a/2}^{a/2} dy' \, V \, \frac{2z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}.$$

The total potential is then:

$$\boxed{\begin{aligned} \underset{(z>0)}{\Phi(\vec{x})} &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right] \\ &+ \frac{Vz}{2\pi}\int_{-a/2}^{a/2} dx' \int_{-a/2}^{a/2} dy' \, \frac{1}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}} \end{aligned}}$$