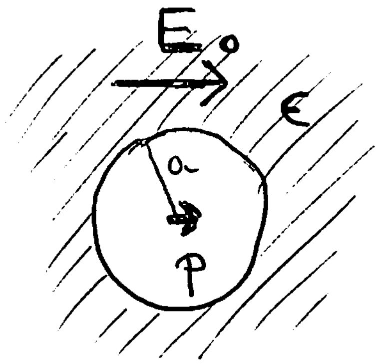
Problem 3



(a)

$$\Phi_{in} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \sum_l A_l r^l P_l(\cos\theta)$$

$$\Phi_{out} = -E_0 r\cos\theta + \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

Note that $\cos\theta = P_1(\cos\theta)$.
Since both the dipole p and the external field E_0 have a $P_1(\cos\theta)$ dependence, then $l=1$ is the only channel of response.

Let us request the continuity of the potential, focusing on $l=1$:

$$\boxed{\frac{p}{4\pi\epsilon_0 a^2} + A_1 a = -E_0 a + \frac{B_1}{a^2}} \qquad (1)$$

We need a second boundary condition.
Let us use $(\vec{D}_2 - \vec{D}_1)\cdot\vec{n}_{21} = 0$.

As discussed in class, this translates into:

$$\epsilon \left.\frac{\partial \Phi_{out}}{\partial r}\right]_{r=a} = \epsilon_0 \left.\frac{\partial \Phi_{in}}{\partial r}\right]_{r=a}$$

$$\epsilon\left[-E_0 \underbrace{\left.\frac{\partial (r)}{\partial r}\right|_{r=a}}_{=1} \cos\theta + B_1 \underbrace{\left.\frac{\partial\left(\frac{1}{r^2}\right)}{\partial r}\right|_{r=a}}_{\left(-\frac{2}{a^3}\right)} P_1(\cos\theta)\right] =$$

$$= \epsilon_0\left[\frac{p}{4\pi\epsilon_0} \underbrace{\left.\frac{\partial\left(\frac{1}{r^2}\right)}{\partial r}\right]_{r=a}}_{\left(-\frac{2}{a^3}\right)} \cos\theta + A_1 \underbrace{\left.\frac{\partial (r)}{\partial r}\right]_{r=a}}_{=1} P_1(\cos\theta)\right]$$

$$\boxed{\epsilon\left[-E_0 - \frac{2B_1}{a^3}\right] = \epsilon_0\left[\frac{-2p}{4\pi\epsilon_0 a^3} + A_1\right]} \quad (2)$$

From ①:

$$A_1 = -E_0 + \frac{B_1}{a^3} - \frac{p}{4\pi\epsilon_0 a^3}$$

Replacing in ②:

$$\epsilon\left[-E_0 - \frac{2B_1}{a^3}\right] = \frac{-2p}{4\pi a^3} + \epsilon_0\left[-E_0 + \frac{B_1}{a^3} - \frac{p}{4\pi\epsilon_0 a^3}\right]$$

$$-\frac{2\epsilon B_1}{a^3} - \frac{\epsilon_0 B_1}{a^3} = \epsilon E_0 - \frac{2p}{4\pi a^3} - \epsilon_0 E_0 - \frac{p}{4\pi a^3}$$

$$-\frac{B_1}{a^3}\left(2\epsilon+\epsilon_0\right) = (\epsilon-\epsilon_0)E_0 - \frac{3p}{4\pi a^3}$$

$$\boxed{B_1 = -a^3\frac{(\epsilon-\epsilon_0)E_0}{(2\epsilon+\epsilon_0)} + \frac{3p}{4\pi(2\epsilon+\epsilon_0)}}$$

$$\boxed{A_1 = -E_0 - \frac{p}{4\pi\epsilon_0 a^3} - \frac{(\epsilon-\epsilon_0)E_0}{(2\epsilon+\epsilon_0)} + \frac{3p}{4\pi a^3(2\epsilon+\epsilon_0)}}$$

$$\boxed{= -E_0\frac{3\epsilon}{(2\epsilon+\epsilon_0)} + \frac{p}{4\pi a^3}\frac{2(\epsilon_0-\epsilon)}{(2\epsilon+\epsilon_0)\epsilon_0}.}$$

$$\boxed{\begin{aligned}\Phi_{in} &= \frac{p\cos\theta}{4\pi\epsilon_0 r^2} + \left[-\frac{E_0 3\epsilon}{(2\epsilon+\epsilon_0)} + \frac{2p(\epsilon_0-\epsilon)}{4\pi a^3(2\epsilon+\epsilon_0)\epsilon_0}\right] r\cos\theta \\ \Phi_{out} &= -E_0 r\cos\theta + \left[\frac{-a^3E_0(\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)} + \frac{3p}{4\pi(2\epsilon+\epsilon_0)}\right]\frac{1}{r^2}\cos\theta\end{aligned}}$$

(b) If $\epsilon = \epsilon_0$, then:

$$\Phi_{in} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} - E_0 r\cos\theta$$

$$\Phi_{out} = -E_0 r\cos\theta + \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

equal and just the sum of dipole + E_0.

(c) The dipole term in Φ_{out} is:

$$\frac{1}{(2\epsilon+\epsilon_0)}\left[-a^3E_0(\epsilon-\epsilon_0) + \frac{3p}{4\pi}\right]\frac{\cos\theta}{r^2}$$

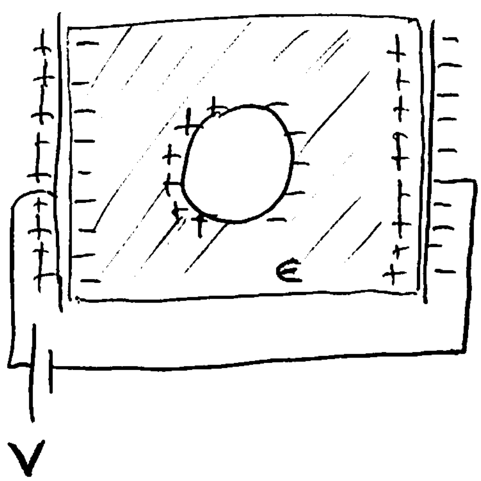
It cancels if $\boxed{\frac{3p}{4\pi} = a^3E_0(\epsilon-\epsilon_0)}$

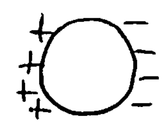
If $p=0$, the diple in Φ_{out} caused by E_0 is

$$-\frac{a^3 E_0 (\epsilon-\epsilon_0)}{(2\epsilon+\epsilon_0)}\cdot\frac{\cos\theta}{r^2}$$

i.e. the effective dipole is negative.

Intuitively this is correct based on intuition:





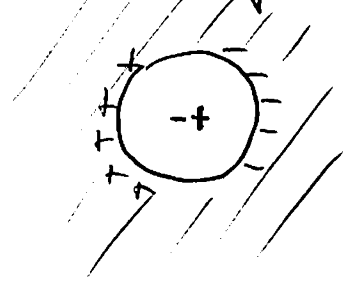
has a dipole pointing to the left

compatible with its negative value from math.

If $E_0=0$, the dipole in Φ_{out} is $\dfrac{3p}{4\pi(2\epsilon+\epsilon_0)}\cdot\dfrac{\cos\theta}{r^2}$,

which is smaller than $\dfrac{p\cos\theta}{4\pi\epsilon_0 r^2}$ i.e. the bare dipole, because $\epsilon>\epsilon_0$.

Intuitively:



i.e. the medium provides a polarization similar to E_0 but it is enough only to reduce the net dipole in Φ_{out}

from $\underset{\text{in } \Phi_{in}}{p} \longrightarrow \underset{\text{in } \Phi_{out}}{p[3\epsilon_0/(2\epsilon+\epsilon_0)]}$

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