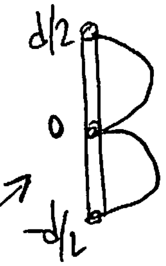
(a) $\vec{J}(\vec{x}') = I \sin\left(\frac{kd}{2} - k|z'|\right)\delta(x')\delta(y')\hat{e}_z$

If $d = \lambda$; $\frac{kd}{2} = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$

then $k|z'| = \frac{2\pi}{\lambda}|z'| = 2\pi\frac{|z'|}{d}$



$$\vec{J}(\vec{x}') = I \sin\left[\pi\left(1 - \frac{2|z'|}{d}\right)\right]\delta(x')\delta(y')\hat{e}_z$$

$$\nabla\cdot\vec{J} = I\,\delta(x')\delta(y')\underbrace{\frac{d}{dz}\sin\left(\pi - 2\pi\frac{|z'|}{d}\right)}_{\begin{cases}\cos\left(\pi - 2\pi\frac{|z'|}{d}\right)\cdot\left(\frac{-2\pi}{d}\right), & z'>0\\ \cos\left(\pi - 2\pi\frac{|z'|}{d}\right)\cdot\left(\frac{+2\pi}{d}\right), & z'<0\end{cases}}$$

$$\nabla\cdot\vec{J} = +i\omega\rho(\vec{x}')$$

$$\rho(\vec{x}') = \frac{(-i)}{\omega} I\,\delta(x')\delta(y')\cos\left(\pi - 2\pi\frac{|z'|}{d}\right)\times\begin{cases}-2\pi/d, & z'>0\\ +2\pi/d, & z'<0\end{cases}$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')\,d^3x' = \int_0^{d/2} z'\frac{(-i)I}{\omega}\cos\left(\pi - \frac{2\pi z'}{d}\right)\left(\frac{-2\pi}{d}\right)dz'$$

along $\hat{e}_z$

$$+\int_{-d/2}^{0} z'\frac{(-i)I}{\omega}\cos\left(\pi + \frac{2\pi z'}{d}\right)\left(+\frac{2\pi}{d}\right)dz' =$$

$$= +\frac{iI\pi 2}{\omega d}\int_0^{d/2} z'\cos\left(\pi-\frac{2\pi z'}{d}\right)dz' - \underbrace{\frac{iI\pi 2}{\omega d}\int_{-d/2}^{0} z'\cos\left(\pi+\frac{2\pi z'}{d}\right)dz'}_{\substack{z'=-u \\ dz'=-du}}$$

$$\int_{-d/2}^{0} z'\cos\left(\pi+\frac{2\pi z'}{d}\right)dz' = \int_{d/2}^{0}(-u)\cos\left(\pi-\frac{2\pi u}{d}\right)(-du)$$

$$= -\int_0^{d/2} u\cos\left(\pi-\frac{2\pi u}{d}\right)du$$

$$= +\frac{4iI\pi}{\omega d}\int_0^{d/2} z'\cos\underbrace{\left(\pi-\frac{2\pi z'}{d}\right)}dz' = +\frac{4iI\pi}{\omega d}\int_{\pi}^{0}\frac{d}{2\pi}(\pi-s)\cos(s)\left(-\frac{d}{2\pi}ds\right)$$

$$s = \pi-\frac{2\pi}{d}z'$$

$$ds = -\frac{2\pi}{d}dz'$$

$$z' = \frac{d}{2\pi}(\pi-s)$$

$$= -\frac{iI}{\omega}\frac{d}{\pi}\int_{\pi}^{0}(\pi-s)\cos s\, ds =$$

$$= \frac{iId}{\omega}\left(+\int_0^{\pi}\cos s\, ds\right) - \frac{iId}{\omega\pi}\int_0^{\pi} s\cos s\, ds =$$

$$= +\frac{iId}{\omega}\left[\left[\sin(s)\right]_0^{\pi}\right] - \frac{iId}{\omega\pi}\underbrace{\left[\left[s\sin(s)\right]_0^{\pi} - \int_0^{\pi}\sin(s)\,ds\right]}_{\text{done by parts}}$$

$$0 + \cos(s)\Big]_0^{\pi} = -2$$

$$= \frac{2iId}{\omega\pi} \quad ; \quad |\vec{p}|^2 = \frac{4I^2d^2}{\omega^2\pi^2}$$

In this case : $k = \frac{2\pi}{d}$, $\frac{c}{\omega} = \frac{1}{k}$

$$\frac{dP}{d\Omega} = \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\cdot\frac{4I^2d^2}{\omega^2\pi^2}\cdot\sin^2\theta =$$

$$\frac{c^2k^4}{\omega^2} = k^2 = \frac{4\pi^2}{d^2}$$

$$= \frac{1}{\cancel{32}_{2}\cancel{\pi^2}}\,\frac{\cancel{4\pi^2}}{\cancel{d^2}}\,\frac{\cancel{4}I^2\cancel{d^2}}{\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\sin^2\theta$$

$$= \boxed{\frac{I^2}{2\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\sin^2\theta = \frac{dP}{d\Omega}}$$

(b) Use Eq. (9.57) for $d = \lambda$

i.e. $k = \frac{2\pi}{\lambda} = \frac{2\pi}{d}$, $kd = 2\pi$.

$$\frac{dP}{d\Omega} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{I^2}{8\pi^2} \, 4 \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

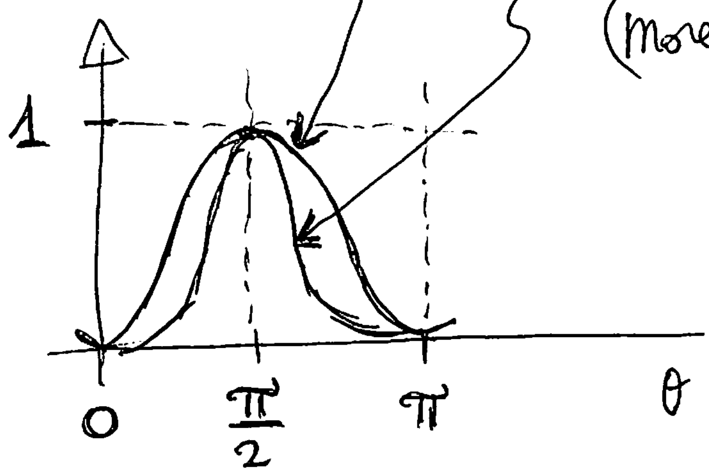
So we need to compare

$$\frac{\sin^2\theta}{2\pi^2} \quad \text{vs.} \quad \frac{1}{2\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

or

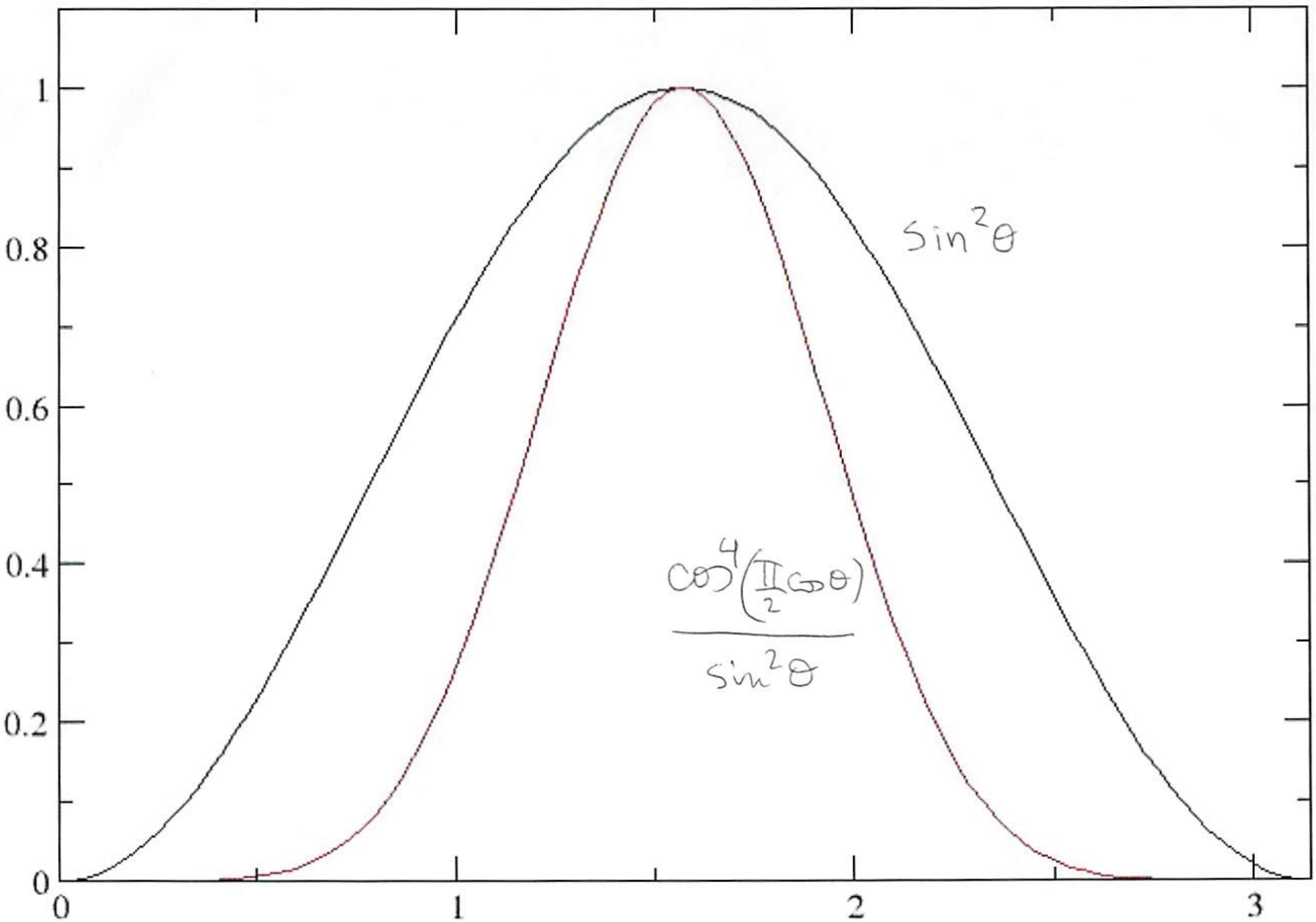
$$\sin^2\theta \quad \text{vs.} \quad \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

(more narrow)



you can use

https://www.desmos.com/calculator



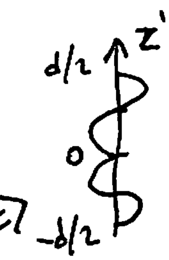
(c) Consider now $d = 2\lambda$:

$$\frac{kd}{2} = \frac{2\pi}{\lambda}\frac{d}{2} = \frac{2\pi}{\lambda}\frac{2\lambda}{2} = 2\pi$$

$$k|z'| = \frac{2\pi}{\lambda}|z'| = \frac{2\pi}{d/2}|z'| = \frac{4\pi}{d}|z'|$$

$$\vec{J}(\vec{x}') = I \underbrace{\sin\left(2\pi - \frac{4\pi}{d}|z'|\right)}_{-\sin\left(\frac{4\pi}{d}|z'|\right)} \delta(x')\delta(y')\,\hat{e}_z$$

cancels at $|z'| = 0, \frac{d}{4}, \frac{d}{2}$ $\Rightarrow$



$$\nabla\cdot\vec{J} = -I\,\delta(x')\delta(y')\frac{d}{dz}\sin\left(\frac{4\pi|z'|}{d}\right) = -\frac{I4\pi}{d}\,\delta(x')\delta(y')\begin{cases}\cos\left(\frac{4\pi z'}{d}\right) & z'>0\\ -\cos\left(-\frac{4\pi z'}{d}\right) & z'<0\end{cases}$$

$$\rho(\vec{x}') = \frac{I4\pi}{d\,i\omega}\,\delta(x')\delta(y')\begin{cases}-\cos\left(\frac{4\pi z'}{d}\right) & z'>0\\ \cos\left(\frac{4\pi z'}{d}\right) & z'<0\end{cases}$$

$$\vec{p} = \int \vec{x}\,\rho(\vec{x}')\,d^3x' \underset{\text{along } \hat{e}_z}{=} \int_0^{d/2} z'\frac{(-i)I}{\omega d}4\pi\left[-\cos\left(\frac{4\pi z'}{d}\right)\right]dz' + \int_{-d/2}^{0} z'\frac{(-i)I}{\omega d}4\pi\left(+\cos\left(\frac{4\pi z'}{d}\right)\right)dz'$$

$$= \frac{-i\,I4\pi}{\omega d}\left[-\int_0^{d/2} z'\cos\left(\frac{4\pi z'}{d}\right)dz' + \int_{-d/2}^{0} z'\cos\left(\frac{4\pi z'}{d}\right)dz'\right]$$

$$\int_{-d/2}^{0} z' \cos\left(\frac{4\pi z'}{d}\right) dz' \underset{z'=-u}{=} \int_{d/2}^{0} (-u) \cos\left(\frac{4\pi(-u)}{d}\right)(-du) =$$

$$= \int_{d/2}^{0} u \cos\left(\frac{4\pi u}{d}\right) du = -\int_{0}^{d/2} u \cos\left(\frac{4\pi u}{d}\right) du$$

$$\vec{p} = \hat{e}_z \times \frac{i\,4\pi I}{\omega d}\, 2 \int_{0}^{d/2} z' \cos\underbrace{\left(\frac{4\pi z'}{d}\right)}_{\frac{4\pi z'}{d} = u} dz' =$$

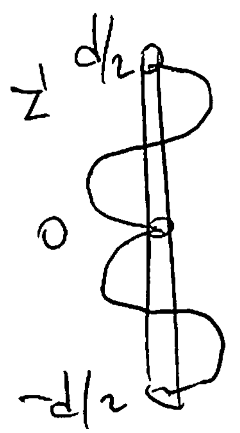
$$= \frac{i\,\cancel{4\pi} I}{\omega \cancel{d}}\, \cancel{2} \int_{0}^{2\pi} \frac{\cancel{d}u}{\cancel{4\pi}} \cos(u) \left(\frac{d}{\cancel{4}\pi}\right)_{2} du = \frac{iI}{\omega 2\pi} \underbrace{\int_{0}^{2\pi} u \cos u \, du}_{\text{by parts}}$$

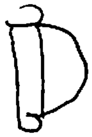
$$\left(\int_a^b v \cos v \, dv = v \sin v \Big]_a^b + \cos v \Big|_a^b \right)$$

$$= \frac{iI}{\omega 2\pi} \left[\underbrace{u \sin u \Big|_0^{2\pi}}_{=0} + \underbrace{\cos u \Big|_0^{2\pi}}_{1-1=0} \right] = 0$$

Then, $\vec{q} = 0$ for this particular current.

The reason is that the current is like:



As explained before each  is like a dipole because what matters is the sign of the slope from $\nabla \cdot \vec{J} = i\omega\rho$.

Then this is like 4 dipoles as:

2 pointing up
2 pointing down

The net dipole moment cancels. What survives is a quadrupolar moment but we will not calculate it in this problem. But intuitively the angular dependence of the radiation pattern should be