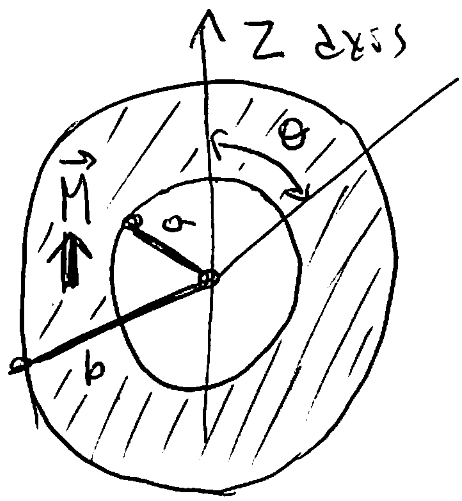
Problem 1



$\vec{M} = M_0 \hat{e}_z$

(a) From lectures and book, it is clear that there are two sources of Φ_M. One of them at the surface of radius b and the other at the surface of radius a. From the example of a hard ferromagnet solved in class we know that at surface "b" there is an effective magnetic surface charge density σ_M of value $\sigma_M^b = \vec{n} \cdot \vec{M}$ where $\vec{n}$ is $\hat{e}_r$.

For the surface "a" we need to be careful. The boundary condition says in general

$$\sigma_M = -(\vec{M}_2 - \vec{M}_1) \cdot \vec{n}_{21}$$

For "a", $\vec{M}_2 = \vec{M}$ and $\vec{M}_1 = 0$ so we get $\sigma_M^a = -\vec{n} \cdot \vec{M}$, with $\vec{n} = \hat{e}_r$. Thus, there is a crucial sign difference between the two.

(b) From the example solved in class, and since $\nabla \cdot \vec{M} = 0$ inside the hard ferromagnet, then:

$$\Phi_M(\vec{x}) = \frac{1}{4\pi} \oint_{S_b} \frac{\hat{e}_r' \cdot \vec{M}\, ds'}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi} \oint_{S_a} \frac{\hat{e}_r' \cdot \vec{M}\, ds'}{|\vec{x}-\vec{x}'|}$$

S_b ↗ sphere radius b S_a ↗ sphere radius a

Also $\hat{e}_r' \cdot \vec{M} = M_0 \cos\theta'$. Then,

$$\Phi_M(\vec{x}) = \frac{b^2}{4\pi} \int_{S_b} \frac{M_0 \cos\theta' d\Omega'}{|\vec{x}-\vec{x}'|} - \frac{a^2}{4\pi} \int_{S_a} \frac{M_0 \cos\theta' d\Omega'}{|\vec{x}-\vec{x}'|}$$

Let us now use the "magic" formula (3.70)

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$r_<$ = Smaller of $|\vec{x}|$ and $|\vec{x}'|$
$r_>$ = larger of $|\vec{x}|$ and $|\vec{x}'|$

Also $\cos\theta' = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\phi')$

$$\Phi_M(\vec{x}) = \frac{b^2}{\cancel{4\pi}} \cancel{4\pi} M_0 \sum_{l,m} \frac{1}{2l+1} \underbrace{\frac{r_<^l}{r_>^l}}_{\text{in } S_b} \int d\Omega' Y_{lm}^*(\theta',\phi') \sqrt{\frac{4\pi}{3}} Y_{10}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$- \frac{a^2}{\cancel{4\pi}} \cancel{4\pi} M_0 \sum_{l,m} \frac{1}{2l+1} \underbrace{\frac{r_<^l}{r_>^{l+1}}}_{\text{in } S_a} \int d\Omega' Y_{lm}^*(\theta',\phi') \sqrt{\frac{4\pi}{3}} Y_{10}(\theta',\phi') Y_{lm}(\theta,\phi)$$

Use $\int d\Omega' \, Y_{l'm'}^*(\theta',\phi') Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$

in general.

Then:

$$\Phi_M \underset{\substack{l=1\\m=0}}{\overset{\uparrow}{=}} \frac{b^2 M_0}{3} \underbrace{\frac{r_<^1}{r_>^2}}_{\text{in } S_b} \underbrace{\sqrt{\frac{4\pi}{3}} Y_{10}(\theta,\phi)}_{\cos\theta} - \frac{a^2 M_0}{3} \underbrace{\frac{r_<^1}{r_>^2}}_{\text{in } S_a} \underbrace{\sqrt{\frac{4\pi}{3}} Y_{10}(\theta,\phi)}_{\cos\theta}$$

$$\Phi_M = \frac{M_0 \cos\theta}{3} \left(b^2 \underbrace{\frac{r_<^1}{r_>^2}}_{\text{in } S_b} - a^2 \underbrace{\frac{r_<^1}{r_>^2}}_{\text{in } S_a} \right).$$

Consider the three regions separately:

Inside i.e. $r \leq a$:

$r_< = r$ in both terms

$r_> = \begin{cases} a & \text{for } S_a \\ b & \text{for } S_b \end{cases}$

$$\Phi_M = \frac{M_0}{3}\cos\theta \, r - \frac{M_0}{3}\cos\theta \, r = \boxed{0}.$$

The contributions of the two spheres cancel out. This probably is natural since the sources had different signs but the same magnitude $\vec{n}\cdot\vec{M}$, there is total shielding.

Intermediate region $a \leq r \leq b$:

For S_a: $r_< = a$, $r_> = r$ For S_b: $r_< = r$, $r_> = b$

$$\Phi_M = \frac{M_0}{3}\cos\theta \, r - \frac{M_0}{3}\cos\theta \, a^2 \frac{a}{r^2} = \boxed{\frac{M_0}{3}\cos\theta\left(r - \frac{a^3}{r^2}\right)}$$

linearly growing Φ_M caused by inner surface

dipole like contribution

In the outside region $r \geq b$:

For S_a: $r_< = a$; For S_b: $r_< = b$

$r_> = r$ $\qquad$ $r_> = r$

$$\Phi_M = \frac{M_0 \cos\theta}{3}\left(\frac{b^3}{r^2} - \frac{a^3}{r^2}\right) = \boxed{\frac{M_0 \cos\theta}{3}\,\frac{(b^3 - a^3)}{r^2}}$$

dipole like

Since the volume magnetized is $\frac{4\pi(b^3-a^3)}{3}$ the "b^3-a^3" factor has a clear origin; it simply reflects the volume that is ferromagnetic.

Is it continuous?

Φ inside at $r=a$ is 0.

Φ intermediate at $r=a$ is 0 because $\left. r - \frac{a^3}{r^2}\right|_{r=a} = 0$.

Φ intermediate at $r=b$ is $\frac{M_0}{3}\cos\theta\left(b - \frac{a^3}{b^2}\right)$

Φ outside at $r=b$ is $\frac{M_0}{3}\cos\theta\left(\frac{b^3-a^3}{b^2}\right)$ which is the same as from Φ intermediate.

Then, Φ_M is continuous.

(c) Consider the z axis i.e. $\theta = 0$:

$$\begin{cases} \Phi_M = 0 \ , & \text{inside } r \leq a \\ \Phi_M = \frac{M_0}{3}\left(z - \frac{a^3}{z^2}\right), & a \leq r \leq b \\ \Phi_M = \frac{M_0}{3}\frac{(b^3-a^3)}{z^2} \ , & r \geq b \end{cases}$$

$$\begin{cases} H_z = 0 & \text{inside } r \leq a \\ H_z = -\nabla \Phi_M = -\frac{d}{dz}\left[\frac{M_0}{3}\left(z - \frac{a^3}{z^2}\right)\right], & a \leq r \leq b \\ H_z = -\frac{d}{dz}\left(\frac{M_0}{3}\frac{(b^3-a^3)}{z^2}\right), & r \geq b \end{cases}$$

$$\begin{cases} H_z = \boxed{0} & \text{inside} \\ H_z = -\frac{M_0}{3} + \frac{M_0}{3} a^3 \left(-\frac{2}{z^2}\right) = \boxed{-\frac{M_0}{3} - 2\frac{M_0}{3}\frac{a^3}{z^3}} \\ H_z = \boxed{\frac{M_0}{3}(b^3-a^3)\frac{2}{z^3}} \end{cases}$$

$$\begin{cases} B_z = 0 & \text{inside} \\ B_z = \mu_0 (H_z + M_z) & \text{intermediate} \\ B_z = \mu_0 H_z & \text{outside} \end{cases}$$

$$B_z = \boxed{0} \quad \text{inside}$$

$$B_z = \mu_0 \left(-\frac{M_0}{3} - 2\frac{M_0}{3}\frac{a^3}{z^3} \right) + \mu_0 M_0 \qquad \text{intermediate}$$

$$= \frac{2}{3}\mu_0 M_0 - \frac{2}{3}\mu_0 M_0 \frac{a^3}{z^3} = \boxed{\frac{2}{3}\mu_0 M_0 \left(1 - \frac{a^3}{z^3}\right)}$$

$$B_z = \boxed{\frac{\mu_0 M_0 2}{3} \frac{(b^3 - a^3)}{z^3}} \qquad \text{outside}$$

<u>Regarding B_z:</u> at $r = a$, $B_z = 0$ intermediate

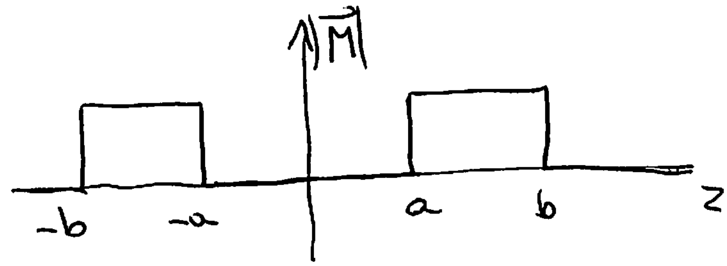
at $r = b$, $B_z = \frac{2}{3}\mu_0 M_0 \left(1 - \frac{a^3}{b^3}\right)$

in both intermediate and outside.

So, as expected, <u>B_z is continuous.</u>

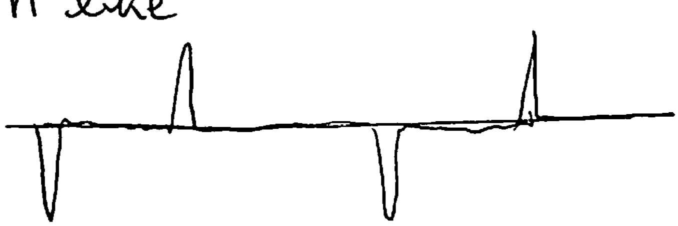
Regarding H_z: at $r=a$, $H_z=0$ inside but $H_z=-M_0$ from result of H_z in intermediate region. Then, H_z is discontinuous at $r=a$ which is reasonable since "$r=a$" is a source of Φ_M and thus $\vec{H}$. At $r=b$, H_z intermediate is $\left(-\frac{M_0}{3}-\frac{2}{3}M_0\frac{a^3}{b^3}\right)$ while H_z outside is $\frac{2M_0}{3}\left(\frac{b^3-a^3}{b^3}\right)$. Then, at $r=b$ H_z is not continuous either, again because H_z has a source at $r=b$.

(d) To sketch $\vec{H}$ remember that "$-\nabla\cdot\vec{M}$" is like "ρ" of the Ch 1 Poisson Equation. If we sketch the profile of M along the z axis we get

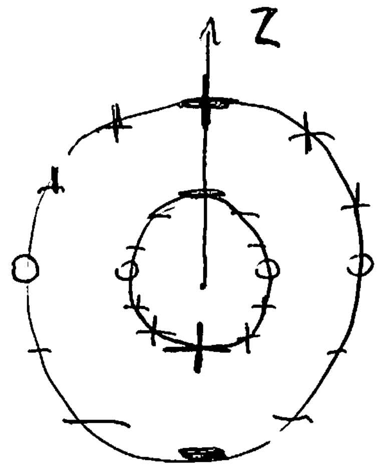


Then $\nabla\cdot\vec{M}$ will look like

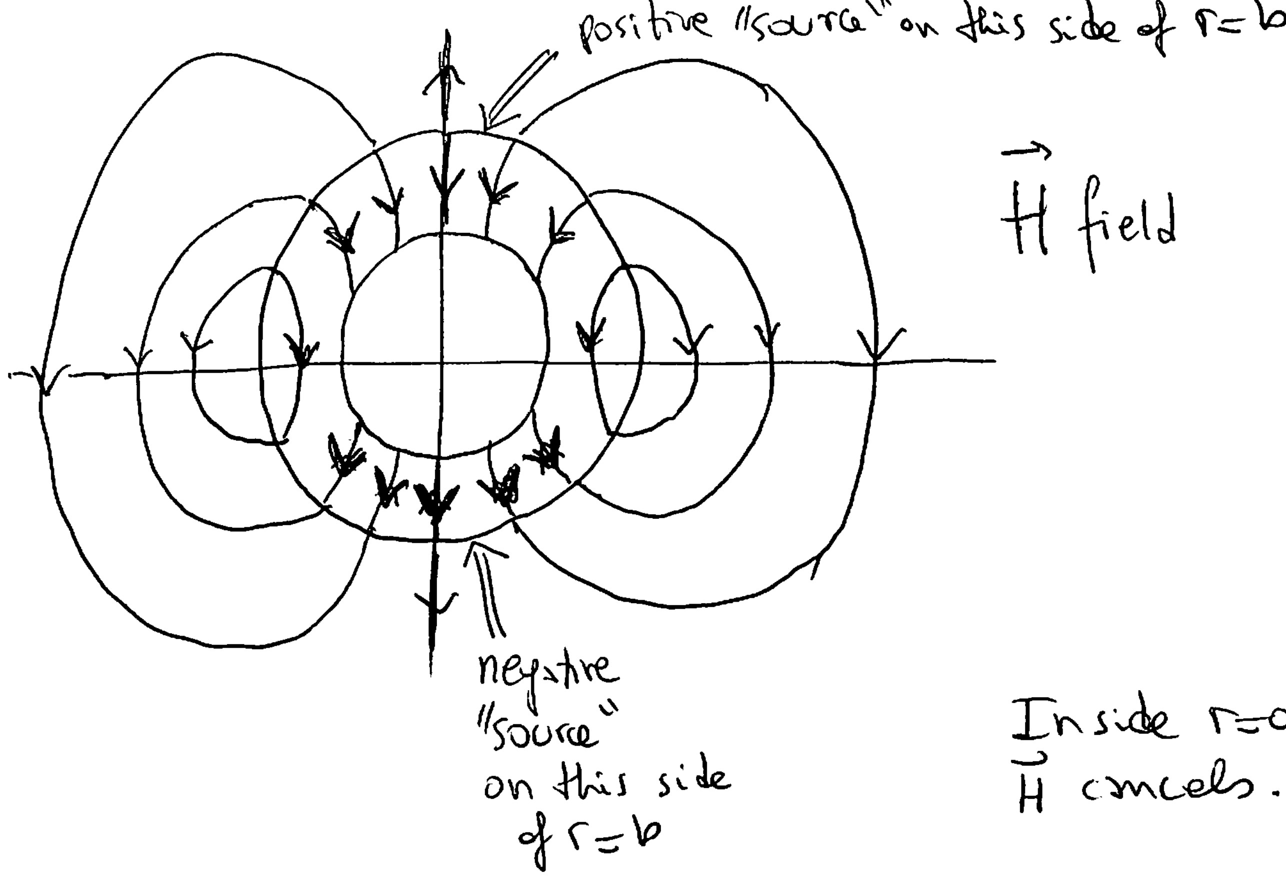
and $-\nabla\cdot\vec{M}$ like



Then, the signs of the effective charge "$-\nabla \cdot \vec{M}$" are like

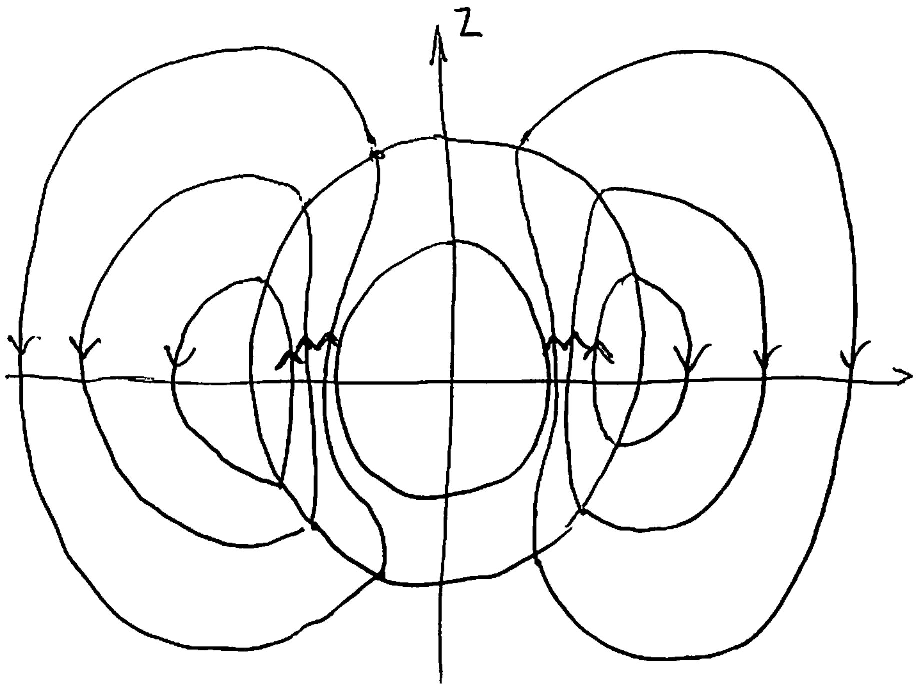


and the fields will be like



Inside $r=a$, $\vec{H}$ cancels.

For the $\vec{B}$ field, I cannot have sources because $\nabla \cdot \vec{B} = 0$ at every single point. Outside it is just proportional to $\vec{H}$. Inside it is 0. The big difference with $\vec{H}$ is in the intermediate region.



$\vec{B}$ field