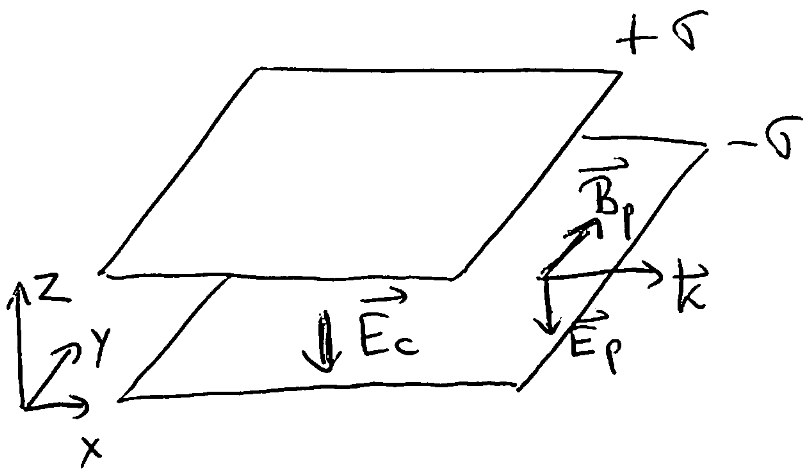
Problem 2



$\vec{E}_c = -E_c \hat{e}_z$ is the electric field caused by the capacitor, which is known to be $E_c = \frac{\sigma}{\epsilon_0}$.

In addition we assume that the wavelength λ of the wave is much larger than the capacitor's length

so that $k = \frac{\omega}{c} \sim \frac{1}{\lambda} \approx 0$.

Then $\vec{E}_p \simeq E_0 \cos(\omega t)\,(-\hat{e}_z)$

$\vec{B}_p \simeq \frac{E_0}{c} \cos(\omega t)\,(+\hat{e}_y)$

The total fields then are

$$\vec{E} = -[E_c + E_0 \cos(\omega t)]\,\hat{e}_z$$
$$\vec{B} = \frac{E_0}{c} \cos(\omega t)\,\hat{e}_y$$

In the notation of Maxwell's stress tensor

$$E_1 = E_2 = 0\,, \quad E_3 \neq 0$$
$$B_1 = B_3 = 0\,, \quad B_2 \neq 0$$

If $\alpha \neq \beta$, then $T_{\alpha\beta} = 0$ in this case.

$$T_{11} = \epsilon_0 \left(-\frac{1}{2}\right)\left(E_3^2 + c^2 B_2^2\right)$$

$$T_{22} = \epsilon_0 \left[c^2 B_2^2 - \frac{1}{2}\left(E_3^2 + c^2 B_2^2\right)\right] = \frac{\epsilon_0}{2}\left(c^2 B_2^2 - E_3^2\right)$$

$$T_{33} = \epsilon_0 \left[E_3^2 - \frac{1}{2}\left(E_3^2 + c^2 B_2^2\right)\right] = \frac{\epsilon_0}{2}\left(E_3^2 - c^2 B_2^2\right)$$

$$= -T_{22}$$

Then:

$$E_3 = -\left[E_c + E_0 \cos(\omega t)\right]$$

$$c B_2 = E_0 \cos(\omega t)$$

$$E_3^2 = E_c^2 + E_0^2 \cos^2(\omega t) + 2 E_c E_0 \cos(\omega t)$$

$$c^2 B_2^2 = E_0^2 \cos^2(\omega t)$$

$$E_3^2 + c^2 B_2^2 = E_c^2 + 2 E_0^2 \cos^2(\omega t) + 2 E_c E_0 \cos(\omega t)$$

$$E_3^2 - c^2 B_2^2 = E_c^2 + 2 E_c E_0 \cos(\omega t)$$

$$T = -\frac{\epsilon_0}{2} \begin{pmatrix} E_c^2 + 2E_0^2\cos^2(\omega t) + 2E_cE_0\cos(\omega t) & 0 & 0 \\ 0 & E_c^2 + 2E_cE_0\cos(\omega t) & 0 \\ 0 & 0 & -\left(E_c^2 + 2E_cE_0\cos(\omega t)\right) \end{pmatrix}$$

If we consider the "time average" then

$$\cos^2(\omega t) \longrightarrow 1/2 \quad (\text{as discussed in class})$$

and

$$\cos(\omega t) \longrightarrow 0$$

$$T = -\frac{\epsilon_0}{2}\begin{pmatrix} E_c^2 + E_0^2 & 0 & 0 \\ 0 & E_c^2 & 0 \\ 0 & 0 & -E_c^2 \end{pmatrix}$$

Force on lower plate (use $\vec{n} = (0,0,+1)$)

$$\frac{F_3}{\text{Area}} = \sum_\beta T_{3\beta} n_\beta = +T_{33} = +\frac{\epsilon_0}{2} E_c^2 = \boxed{+\frac{\sigma^2}{2\epsilon_0}}$$

which is the same that can be found in the absence of the plane wave.

This result makes sense because the $\vec{E}$ field of the plane wave is oscillating from $(+)$ to $(-)$ in the z direction and its effect cancels out.

(b) Consider now the time dependent result:

$\vec{n} = (0,0,+1)$ all the time so F_3 is the only force that matters.

$$\frac{F_3}{\text{Area}} = \sum_\beta T_{3\beta} n_\beta = +T_{33} = \boxed{+\frac{\epsilon_0}{2}\left[E_c^2 + 2E_cE_0\cos(\omega t)\right]}$$

When $\cos(\omega t) = -1$, then

$$\left(E_c = \frac{\sigma}{\epsilon_0}\right)$$

$$\frac{F_3}{\text{Area}} = \frac{\epsilon_0}{2}E_c\left(E_c - 2E_0\right)$$

(c) If $E_0 \gg E_c$ then $\frac{F_3}{\text{Area}} \approx \boxed{-\epsilon_0 E_c E_0}$ is a force that can point down.

Note that the plane wave alone cannot produce a force on the lower plate because its wavevector is parallel to the plate, i.e. if $E_c = 0$ then the stress tensor is

$$T = \begin{pmatrix} -\epsilon_0 E_0^2\cos^2(\omega t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\frac{F_3}{\text{Area}} = \sum_\beta T_{3\beta} n_\beta = 0$ on the plates.

But in the presence of $E_c \neq 0$, (i.e. $\frac{\sigma}{\epsilon_0}$) then there is a time dependent force on the lower plate with an amplitude proportional to E_0.