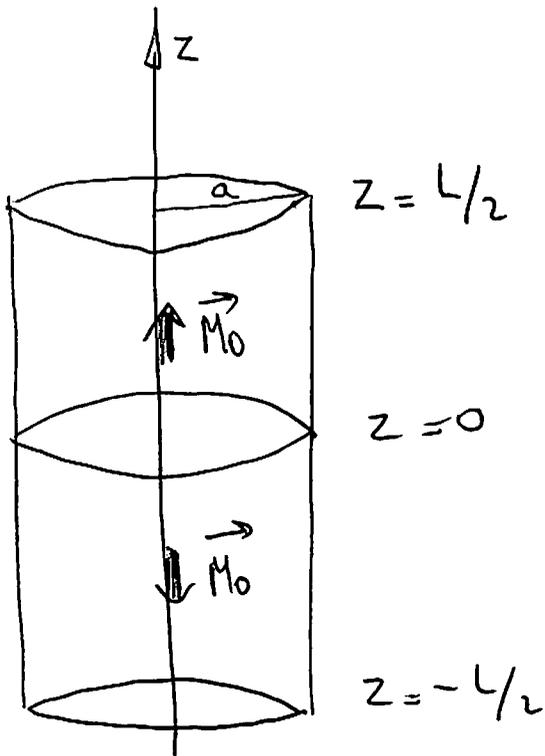


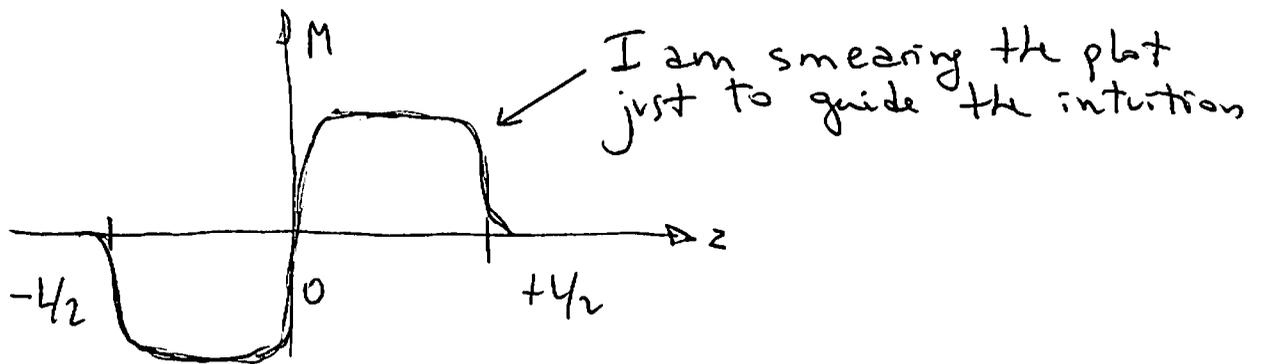
1



(2)

Inside each of the two pieces of the magnet $\nabla \cdot \vec{M} = 0$ since \vec{M} is uniform inside each. Then all the "action" happens at the interfaces. The first step is to identify the sign of each contribution and its magnitude.

Let us make a sketch of the magnetization \vec{M} vs. z :

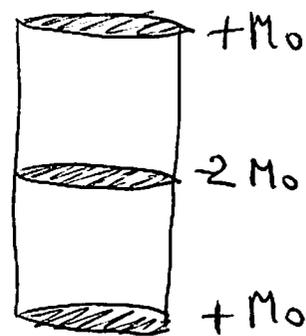


From (5.96), $\rho_M = -\nabla \cdot \vec{M}$, we see that at $z = +L/2$ $\frac{dM}{dz}$ is negative, thus $-\frac{dM}{dz}$ is positive. Then, at $z = +L/2$ there is a positive "magnetic surface-charge density" as explained in (5.99) $\sigma_M = \vec{n} \cdot \vec{M}$

Repeating for $z = -L/2$, $-\frac{dM}{dz}$ is again positive (this is clear from the sketch). However, for $z = 0$, $\frac{dM}{dz}$ is positive and twice as large in magnitude as for $z = \pm L/2$, thus $-\frac{dM}{dz}$ is negative.

In summary:

$$\begin{cases} \text{At } z = +L/2, \sigma_M = +M_0 \\ \text{At } z = 0, \sigma_M = -2M_0 \\ \text{At } z = -L/2, \sigma_M = +M_0 \end{cases}$$



(b)

At this point we simply use (S.100) dropping the volume integral and keeping only the surface integrals.

$$\Phi_M(\vec{x}) = \frac{M_0}{4\pi} \int_{\uparrow \text{top}} \frac{da'}{|\vec{x} - \vec{x}'|} - \frac{2M_0}{4\pi} \int_{\uparrow \text{center}} \frac{da'}{|\vec{x} - \vec{x}'|} + \frac{M_0}{4\pi} \int_{\uparrow \text{bottom}} \frac{da'}{|\vec{x} - \vec{x}'|}$$

$$\text{where } \int da' = \int_0^a \rho' d\rho' \int_0^{2\pi} d\phi'$$

We want to study results along the z axis only.

Thus: $\vec{X} = (0, 0, z)$

$\vec{X}' = (\rho', \phi', \begin{matrix} L/2 \\ 0 \\ -L/2 \end{matrix})$ for the three sources.

Then:

$$|\vec{X} - \vec{X}'| = \begin{cases} \sqrt{\rho'^2 + (\frac{L}{2} - z)^2} \\ \sqrt{\rho'^2 + z^2} \\ \sqrt{\rho'^2 + (-\frac{L}{2} - z)^2} \end{cases}$$

$$\begin{aligned} \Phi_M &= \frac{M_0}{4\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + (\frac{L}{2} - z)^2}} - \frac{2M_0}{4\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + z^2}} + \\ &+ \frac{M_0}{4\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + (-\frac{L}{2} - z)^2}} \end{aligned}$$

$$\Phi_M = \frac{M_0}{2} \cdot \left. \sqrt{\rho^2 + \left(\frac{L}{2} - z\right)^2} \right|_0^a - M_0 \cdot \left. \sqrt{\rho^2 + z^2} \right|_0^a$$

$$+ \frac{M_0}{2} \cdot \left. \sqrt{\rho^2 + \left(\frac{L}{2} + z\right)^2} \right|_0^a$$

There are four regions in the problem:

① $z > L/2$

② $0 < z < L/2$

③ $-L/2 < z < 0$

④ $-L/2 \gg z$

that must be treated separately.

Consider region ①:

$$\Phi_M = \frac{M_0}{2} \left[\sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \sqrt{\left(\frac{L}{2} - z\right)^2} \right]$$

$$- M_0 \left[\sqrt{a^2 + z^2} - \sqrt{z^2} \right]$$

$$+ \frac{M_0}{2} \left[\sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} - \sqrt{\left(\frac{L}{2} + z\right)^2} \right]$$

This formula is valid in all 4 regions.

If $z > L/2$ the three terms without "a" become

$$- \left(z - \frac{L}{2} \right) + 2z - \left(z + \frac{L}{2} \right) = \cancel{L} \cdot 0$$

If $0 < z < L/2$ in region (2):

$$-\left(\frac{L}{2} - z\right) + 2z - \left(\frac{L}{2} + z\right) = 2z - L$$

If $-\frac{L}{2} < z < 0$ in region (3):

$$-\left(\frac{L}{2} - z\right) - 2z - \left(\frac{L}{2} + z\right) = -2z - L$$

If $z < -L/2$ in region (4):

$$-\left(\frac{L}{2} - z\right) - 2z + \left(\frac{L}{2} + z\right) = 0$$

Then:

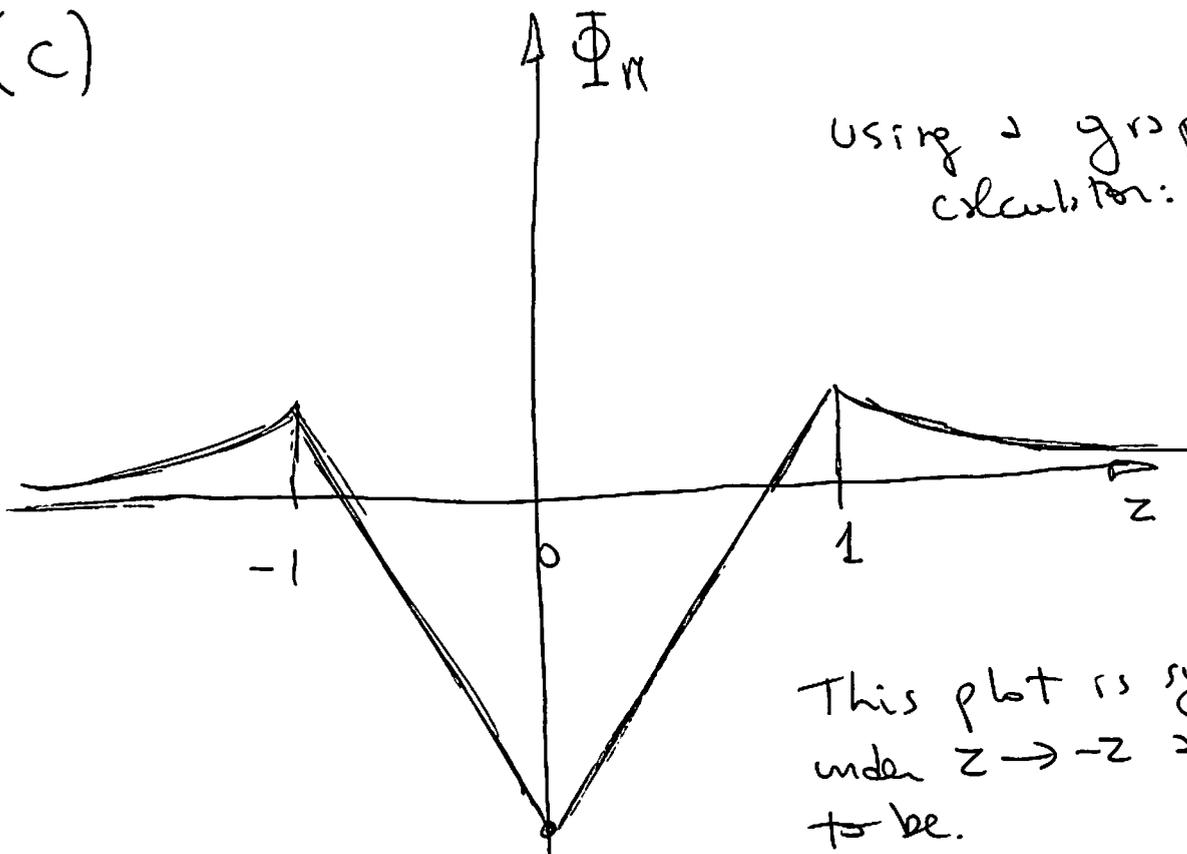
$$\Phi_M = \begin{cases} \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \mu_0 \sqrt{a^2 + z^2} + \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} \\ \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \mu_0 \sqrt{a^2 + z^2} + \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} (2z - L) \\ \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \mu_0 \sqrt{a^2 + z^2} + \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} -(2z + L)\mu_0 \\ \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \mu_0 \sqrt{a^2 + z^2} + \frac{\mu_0}{2} \sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} \end{cases}$$

in the four regions.

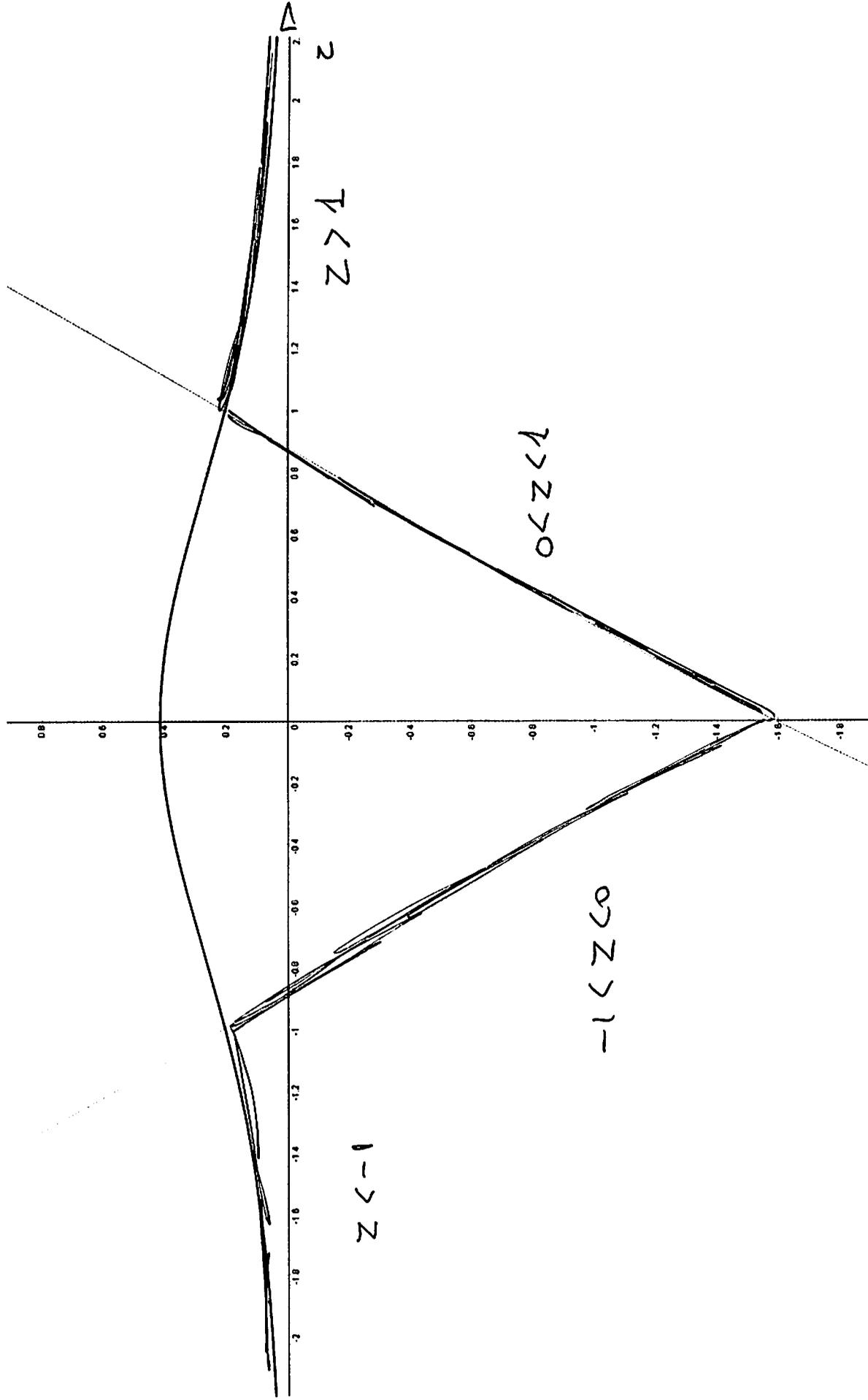
Plot for $M_0 = 1, a = 1, L = 2$:

$$\Phi_M = \begin{cases} \frac{1}{2} \sqrt{1 + (1-z)^2} - \sqrt{1+z^2} + \frac{1}{2} \sqrt{1+(1+z)^2} & , z > 1 \\ \frac{1}{2} \sqrt{1 + (1-z)^2} - \sqrt{1+z^2} + \frac{1}{2} \sqrt{1+(1+z)^2} + 2(z-1) & , 0 < z < 1 \\ \frac{1}{2} \sqrt{1 + (1-z)^2} - \sqrt{1+z^2} + \frac{1}{2} \sqrt{1+(1+z)^2} - 2(z+1) & , -1 < z < 0 \\ \frac{1}{2} \sqrt{1 + (1-z)^2} - \sqrt{1+z^2} + \frac{1}{2} \sqrt{1+(1+z)^2} & , z < -1 \end{cases}$$

(c)



(I used <https://www.desmos.com/calculator>)

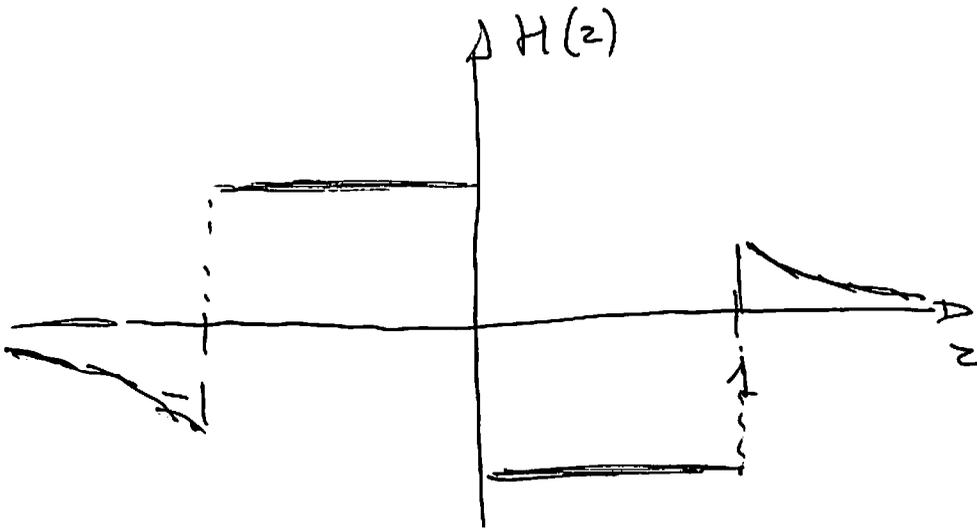


(d)

Using $\vec{H} = -\nabla\Phi_M$, then along the z -axis:

$$H(z) = -\frac{d}{dz}\Phi_M(z)$$

and from the plot of Φ_M , then we can make the sketch:



which now is "odd" under $z \rightarrow -z$

The discontinuities arise from the location of "magnetic surface charge". Since $\vec{H} = -\nabla\Phi_M$ it is naturally discontinuous at $z = -1, 0, +1$.

But \vec{B} should be continuous if we were to calculate it.