

# P541 Electromagnetic Theory I

## First Midterm Exam

February 18, 2011

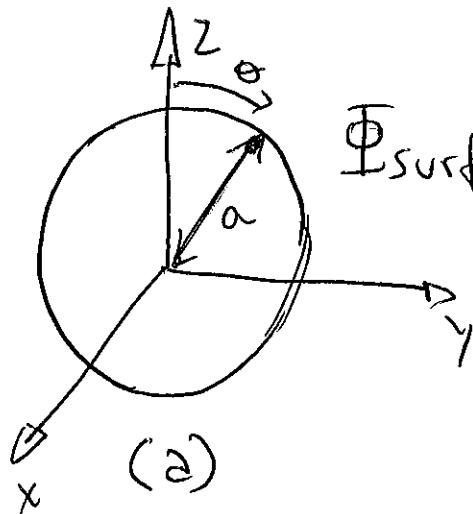
**Problem 1:** A spherical shell of radius  $a$  (assumed to be infinitesimally thin) has a potential fixed to  $V_0 \cos\theta$ , where  $\theta$  is the standard angle used in spherical coordinates.

- (a) Find the potential  $\phi$  everywhere using separation of variables. Both inside and outside the sphere, there is just empty space.
- (b) Find the force on a small test charge  $\delta q$  that we locate at a point  $z_0$  ( $z_0 > a$ ) on the  $z$  axis.

**Problem 2:** In class, following Section 4.4 of the book, we discussed the case of a point charge  $q$  embedded in a semi-infinite dielectric  $\epsilon_1$  a distance  $d$  away from a plane interface that separates the first medium from another semi-infinite dielectric  $\epsilon_2$ . Our main result was Equation (4.47) containing the polarization-charge density at the interface. Repeat the problem, i.e. find the interfacial polarization-charge density, now relabeling  $q$  as  $q_1$  and, more importantly, locating in the medium  $\epsilon_2$  a new charge of magnitude  $q_2$  a distance  $d$  away from the plane interface.  $q_1$  and  $q_2$  are along the  $z$  axis that is perpendicular to the interface. Check that the result reduces to Equation (4.47) when  $q_2=0$  and  $q_1=q$ .

# Problem 1

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$$\Phi_{\text{surface}} = V_0 \cos \theta$$

For  $r < a$ :  $\underbrace{\Phi(r, \theta)}_{\substack{\text{no } \phi \\ \text{dependence} \\ \text{by symmetry}}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$  (no  $\frac{1}{rl+1}$  terms to avoid  $r=0$  divergence)

For  $r > a$ :  $\Phi(r, \theta) = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos \theta)$  (no  $r^l$  terms to avoid  $r \rightarrow \infty$  divergence)

The potential has to be continuous at  $r=0$ :

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos \theta)$$

$$\text{or } B_l = A_l a^{2l+1}$$

In addition we know  $\Phi_{\text{surface}} = V_0 \cos \theta$ . Then,

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = V_0 \cos \theta$$

But  $\cos \theta = P_1(\cos \theta)$ . Multiplying left and right by  $P_m(\cos \theta)$  and integrating we get:

$$\sum_{l=0}^{\infty} A_l a^l \left\{ \int_0^{\pi} d\theta \sin \theta P_m(\cos \theta) P_l(\cos \theta) \right\} = V_0 \times \frac{2}{2m+1} S_{ml}$$

$$V_0 \times \left\{ \int_0^{\pi} d\theta \sin \theta P_m(\cos \theta) P_1(\cos \theta) \right\} = \frac{2}{2m+1} S_{m,1}$$

$$A_m a^m \frac{2}{2m+1} = V_0 \frac{2}{3} S_{m,1}$$

Then,  $A_m = 0$  if  $m \neq 1$

$$A_1 = \frac{2}{3} \frac{V_0}{a} \frac{3}{2} = \frac{V_0}{a} \quad \text{if } m=1$$

Thus :

$$\Phi_{\text{inside}}(r, \theta) = \frac{V_0}{a} r \underbrace{P_1(\cos \theta)}_{\cos \theta} = \boxed{V_0 \left(\frac{r}{a}\right) \cos \theta.}$$

$$\Phi_{\text{outside}}(r, \theta) = \frac{V_0}{a} a^3 \frac{1}{r^2} P_1(\cos \theta)$$

$$= \boxed{V_0 \left(\frac{a}{r}\right)^2 \cos \theta.}$$

(b)  $\vec{F} = q \vec{E}$ . Then, we need  $\vec{E}$  at  $(0, 0, z_0)$ .

$$\vec{E} = -\nabla \Phi$$

"outside"  
Since  $q$  is located outside sphere

$$= -\frac{\partial \Phi}{\partial r} \hat{e}_r - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{e}_\theta =$$

From back of book

$$= -\frac{\partial}{\partial r} \left( V_0 \left(\frac{a}{r}\right)^2 \cos \theta \right) \hat{e}_r - \frac{1}{r} \frac{\partial}{\partial \theta} \left( V_0 \left(\frac{a}{r}\right)^2 \cos \theta \right) \hat{e}_\theta$$

$$= -V_0 a^2 \cos \theta \left(-2 \frac{1}{r^3}\right) \hat{e}_r - \frac{1}{r} V_0 \left(\frac{a}{r}\right)^2 (-\sin \theta) \hat{e}_\theta$$

We are interested in  $(0, 0, z_0)$  which means  
 $r = z_0, \theta = 0$ . Then:

$$\vec{E} = \frac{2V_0 a^2}{z_0^3} \hat{e}_z$$

$$\vec{F} = \frac{2q V_0 a^2}{z_0^3} \hat{e}_z$$