

P541 Electromagnetic Theory I

First Midterm Exam

February 18, 2011

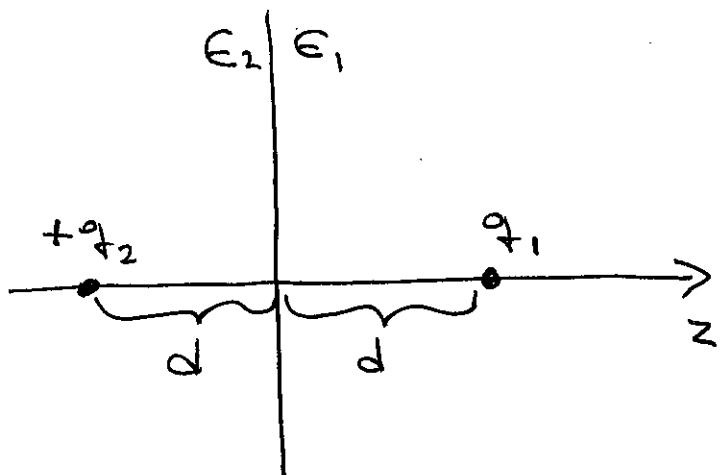
Problem 1: A spherical shell of radius a (assumed to be infinitesimally thin) has a potential fixed to $V_0 \cos\theta$, where θ is the standard angle used in spherical coordinates.

- (a) Find the potential ϕ everywhere using separation of variables. Both inside and outside the sphere, there is just empty space.
- (b) Find the force on a small test charge δq that we locate at a point z_0 ($z_0 > a$) on the z axis.

Problem 2: In class, following Section 4.4 of the book, we discussed the case of a point charge q embedded in a semi-infinite dielectric ϵ_1 a distance d away from a plane interface that separates the first medium from another semi-infinite dielectric ϵ_2 . Our main result was Equation (4.47) containing the polarization-charge density at the interface. Repeat the problem, i.e. find the interfacial polarization-charge density, now relabeling q as q_1 and, more importantly, locating in the medium ϵ_2 a new charge of magnitude q_2 a distance d away from the plane interface. q_1 and q_2 are along the z axis that is perpendicular to the interface. Check that the result reduces to Equation (4.47) when $q_2=0$ and $q_1=q$.

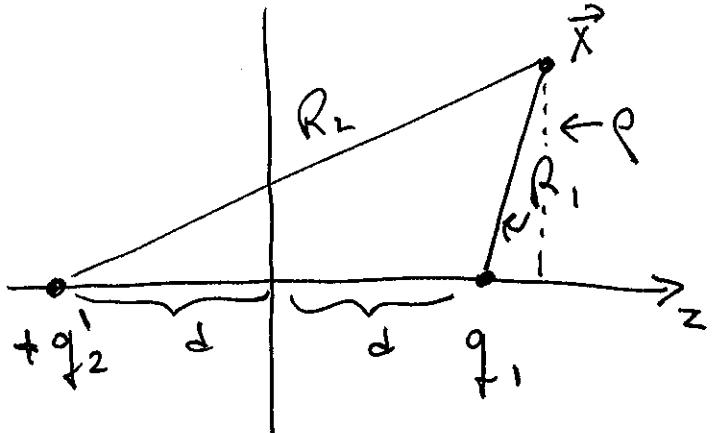
Problem 2

Actual set-up



Consider the perspective of an observer at $z > 0$. She will see the real charge ~~q_1~~ \cancel{q}_1 and an imaginary charge which in general will not be the same as $(+q)_2$. We will locate that charge at $(0, 0, -d)$, like $(+q)_2$ is, but we will give to this charge the value $(+q'_2)$.

Then, the potential will be:



$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_1} \left(\frac{q_1}{R_1} + \frac{q'_2}{R_2} \right)$$

rest + image

$$\text{with } R_1 = \sqrt{\rho^2 + (z-d)^2}$$

$$R_2 = \sqrt{\rho^2 + (z+d)^2}$$

To use the boundary conditions we will need the electric field caused by $\Phi(\vec{x})$ at $z=0$.

$$E_z \Big|_{z=0} = -\frac{d}{dz} \Phi(\vec{x}) \Big|_{z=0} = -\frac{q_1}{4\pi\epsilon_1} \underbrace{\frac{d}{dz} \left(\frac{1}{R_1} \right)}_{z=0} + \frac{q'_2}{4\pi\epsilon_1} \underbrace{\frac{d}{dz} \left(\frac{1}{R_2} \right)}_{z=0}$$

$$\frac{d}{(\rho^2 + d^2)^{3/2}} \quad \frac{-d}{(\rho^2 + d^2)^{3/2}}$$

and the combo " $\epsilon_1 E_z$ " is:

$$\lim_{\substack{z \rightarrow 0 \\ z > 0}} \epsilon_1 E_z = \frac{\epsilon_1 d}{4\pi\epsilon_1 (\rho^2 + d^2)^{3/2}} \left[-q_1 + q'_2 \right].$$

We will also need the tangential component at the plane $z=0$.

$$\lim_{z \rightarrow 0^+} E_\rho = -\frac{d}{d\rho} \Phi(\vec{x}) \Big|_{z=0} = -\frac{q_1}{4\pi\epsilon_1} \underbrace{\frac{d}{d\rho} \left(\frac{1}{R_1} \right)}_{z=0} + \frac{q'_2}{4\pi\epsilon_1} \underbrace{\frac{d}{d\rho} \left(\frac{1}{R_2} \right)}_{z=0}$$

$$\frac{-\rho}{(\rho^2 + d^2)^{3/2}} \quad \frac{-\rho}{(\rho^2 + d^2)^{3/2}}$$

$$= \frac{\rho}{4\pi\epsilon_1 (\rho^2 + d^2)^{3/2}} \left[q_1 + q'_2 \right]$$

Let us repeat the same calculation but for $z < 0$. In this case the real charge is (q_2) , located at $(0, 0, -d)$, and the imaginary charge (or image charge) is $(+q'_1)$ located at $(0, 0, d)$. Note that q'_1 is not equal to q_1 in general.

$$\Phi(z) = \frac{1}{4\pi\epsilon_0} \left(\frac{q'_1}{R_1} + \frac{q_2}{R_2} \right)$$

real + image

$$E^z = -\frac{d}{dz} \left. \Phi(z) \right|_{z=0} = -\frac{q'_1}{4\pi\epsilon_0} \frac{d}{dz} \left. \left(\frac{1}{R_1} \right) \right|_{z=0} - \frac{q_2}{4\pi\epsilon_0} \frac{d}{dz} \left. \left(\frac{1}{R_2} \right) \right|_{z=0}$$

d

$\frac{-d}{(r^2+d^2)^{3/2}}$

and the combo " $\epsilon_0 E^z$ " is:

$$\lim_{\substack{z \rightarrow 0 \\ z < 0}} \epsilon_0 E^z = \frac{\epsilon_0 d}{4\pi\epsilon_0} \frac{1}{(r^2+d^2)^{3/2}} \left[-q'_1 + q_2 \right]$$

For the tangential component:

$$\lim_{z \rightarrow 0^-} E^r = -\frac{d}{dr} \left. \Phi(z) \right|_{z=0} = -\frac{q'_1}{4\pi\epsilon_0} \frac{d}{dr} \left. \left(\frac{1}{R_1} \right) \right|_{z=0} - \frac{q_2}{4\pi\epsilon_0} \frac{d}{dr} \left. \left(\frac{1}{R_2} \right) \right|_{z=0}$$

$\frac{-q}{(r^2+d^2)^{3/2}}$

and we get

$$\lim_{z \rightarrow 0^+} E^{\rho} = \frac{\rho}{4\pi\epsilon_2(\rho^2+d^2)^{3/2}} [q_1' + q_2']$$

The boundary conditions say that

$$\left. \epsilon_1 E^z \right|_{z>0} = \left. \epsilon_2 E^z \right|_{z<0} \quad \text{and} \quad \left. E^{\rho} \right|_{z>0} = \left. E^{\rho} \right|_{z<0}$$

Then:

$$\frac{d}{4\pi (\rho^2+d^2)^{3/2}} (-q_1 + q_2') = \frac{d}{4\pi (\rho^2+d^2)^{3/2}} (-q_1' + q_2)$$

or

$$-q_1 + q_2' = -q_1' + q_2,$$

and the second boundary condition says:

$$\frac{\rho}{4\pi\epsilon_1(\rho^2+d^2)^{3/2}} (q_1 + q_2') = \frac{\rho}{4\pi\epsilon_2(\rho^2+d^2)^{3/2}} (q_1' + q_2)$$

or

$$\frac{q_1 + q_2'}{\epsilon_1} = \frac{q_1' + q_2}{\epsilon_2}.$$

In the special case

$$\begin{aligned}q_1 &= q \\q_2 &= 0 \\q'_2 &= q' \\q''_1 &= q''\end{aligned}$$

we get

$$\left. \begin{aligned}-q + q' &= -q'' + 0 \\ \frac{q + q'}{\epsilon_1} &= \frac{q'' + 0}{\epsilon_2}\end{aligned}\right\}$$

These are the equations found in section 4.4 of the book

Let us solve our equations:

$$q'_2 = q_1 + q_2 - q'_1$$

$$\frac{q_1 + q_1 + q_2 - q'_1}{\epsilon_1} = \frac{q'_1 + q_2}{\epsilon_2}$$

$$2q_1 + q_2 - q'_1 = \frac{\epsilon_1}{\epsilon_2} (q'_1 + q_2)$$

$$2q_1 + q_2 - \frac{\epsilon_1}{\epsilon_2} q_2 = \left(\frac{\epsilon_1}{\epsilon_2} + 1\right) q'_1$$

$$q'_1 = \frac{1}{1 + \frac{\epsilon_1}{\epsilon_2}} \left(2q_1 + q_2 - \frac{\epsilon_1}{\epsilon_2} q_2\right) = \frac{2}{1 + \frac{\epsilon_1}{\epsilon_2}} q_1 + q_2 \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 + \frac{\epsilon_1}{\epsilon_2}}$$

$$q'_2 = q_1 + q_2 - \frac{1}{1 + \frac{\epsilon_1}{\epsilon_2}} \left(2q_1 + q_2 - \frac{\epsilon_1}{\epsilon_2} q_2\right)$$

$$q_2' = q_1 + q_2 - \frac{2}{1 + \frac{\epsilon_1}{\epsilon_2}} q_1 + \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 + \frac{\epsilon_1}{\epsilon_2}} q_2$$

$$= - \left(\frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 + \frac{\epsilon_1}{\epsilon_2}} \right) q_1 + \frac{\frac{2\epsilon_1}{\epsilon_2} q_2}{1 + \frac{\epsilon_1}{\epsilon_2}}$$

To calculate ΔP we need $\Delta \vec{P}$.

$$\Delta \vec{P} = \vec{P}_z^2 - \vec{P}_z' \Big|_{z=0} = \epsilon_0 \left(\frac{\epsilon_2}{\epsilon_0} - 1 \right) E_z^2 - \epsilon_0 \left(\frac{\epsilon_1}{\epsilon_0} - 1 \right) E_z'^2 \Big|_{z=0}$$

$$\frac{\epsilon_1}{\epsilon_2} E_z'$$

$$= \left[\underbrace{\epsilon_0 \left(\frac{\epsilon_2 - \epsilon_0}{\epsilon_0} \right) \frac{\epsilon_1}{\epsilon_2}}_{\epsilon_2 \epsilon_1 - \epsilon_0 \epsilon_1 - \epsilon_2 \epsilon_0 + \epsilon_2 \epsilon_0} - \underbrace{\epsilon_0 \left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_0} \right)}_{\epsilon_1 \epsilon_2} \right] E_z' \Big|_{z=0}$$

$$= \frac{\epsilon_2 \epsilon_1 - \epsilon_0 \epsilon_1 - \epsilon_2 \epsilon_0 + \epsilon_2 \epsilon_0}{\epsilon_2} \frac{d}{4\pi \epsilon_1 (r^2 + d^2)^{3/2}} (q_1 + q_2')$$

$$= \frac{(\epsilon_2 - \epsilon_1) \epsilon_0}{\epsilon_2 \epsilon_1} \frac{d}{4\pi} \frac{1}{(r^2 + d^2)^{3/2}} \underbrace{(q_1 + q_2')}$$

$$-q_1 + \left(\frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 + \frac{\epsilon_1}{\epsilon_2}} \right) q_1 + \frac{\frac{2\epsilon_1}{\epsilon_2} q_2}{1 + \frac{\epsilon_1}{\epsilon_2}}$$

$$-q_1 + q_2' = \frac{-2q_1}{1 + \frac{\epsilon_1}{\epsilon_2}} + \frac{2\epsilon_1/\epsilon_2 q_2}{1 + \frac{\epsilon_1}{\epsilon_2}}$$

$$\sigma_{\text{pol}} = \Delta P = \frac{(\epsilon_2 - \epsilon_1) \epsilon_0}{\epsilon_1 \epsilon_2} \frac{d}{4\pi} \frac{1}{(r^2 + d^2)^{3/2}} \cdot \underbrace{\frac{(-2q_1 + 2\epsilon_1/\epsilon_2 q_2)}{1 + \frac{\epsilon_1}{\epsilon_2}}}_{-2q_1\epsilon_2 + 2\epsilon_1 q_2} \frac{1}{\epsilon_1 + \epsilon_2}$$

$$\boxed{\sigma_{\text{pol}} = \frac{d \cdot \epsilon_0}{4\pi (r^2 + d^2)^{3/2}} \cdot \frac{(\epsilon_2 - \epsilon_1)(-2)(q_1\epsilon_2 - q_2\epsilon_1)}{\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2)}}$$

If $q_1 = q$, $q_2 = 0$ we get $\sigma_{\text{pol}} = -\frac{q}{2\pi} \frac{\epsilon_0}{\epsilon_1} \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} \right) \frac{d}{(r^2 + d^2)^{3/2}}$
which is what we found in Sec. 4.4.

If $\epsilon_1 = \epsilon_2$, then $\sigma_{\text{pol}} = 0$ since there is no interface.

If I switch 1 with 2 everywhere

$$\epsilon_2 - \epsilon_1 \rightarrow -(\epsilon_2 - \epsilon_1)$$

$$q_1\epsilon_2 - q_2\epsilon_1 \rightarrow q_2\epsilon_1 - q_1\epsilon_2 = -(q_1\epsilon_2 - q_2\epsilon_1)$$

Out of two signs, we see that σ_{pol} is invariant
(~) under $1 \rightarrow 2$, $2 \rightarrow 1$
which is correct.

If $\epsilon_1 = \epsilon_2$, then $q_2' = q_2$ and $q_1' = q_1$.