

P541 Electromagnetic Theory I

Final Exam

April 25, 2011

A “rotating” electric dipole can be thought of as the superposition of two oscillating dipoles, one along the x axis, and the other along the y axis, with the latter *out* of phase by 90° :

$$\mathbf{p} = p_0 (\cos\omega t \mathbf{i} + \sin\omega t \mathbf{j}),$$

where \mathbf{i} and \mathbf{j} are the unit vectors along the x and y axis, respectively, ω is the common frequency of oscillation of the two dipoles, and p_0 is a constant. Find the intensity of the radiation $dP/d\Omega$, sketch the intensity profile as a function of the polar angle θ , and calculate the total power radiated. Does the answer seem reasonable?

Let us start with formulas such as

$$\vec{A}(\vec{r}) = -i \frac{\mu_0 \omega}{4\pi} \vec{p} e^{ikr}$$

As discussed in class, we should understand these formulas as always being multiplied by $e^{i\omega t}$ and we can define the ω -dependent dipole (electric) moment as

$$\vec{p}(\omega) = \vec{p} e^{i\omega t}.$$

With regards to the x component of the given polarization, we are set directly since we can use

$$\text{Re } \vec{p}_x(\omega) = \vec{p}_x \cos \omega t \quad \text{and} \quad \vec{p}_x = p_0 \hat{i} \quad (p_0 \in \mathbb{R}).$$

With regards to the other component, we need a moment such that

$$\text{Re } \vec{p}_y(\omega) = \vec{p}_y \sin \omega t, \quad \vec{p}_y = p_0 \hat{j}$$

then $\vec{p}_y(\omega) = (-i) \vec{p}_y e^{i\omega t}$. Thus,

$$\vec{p}_{\text{total}}(\omega) = p_0 \hat{i} e^{i\omega t} + (-i) p_0 \hat{j} e^{i\omega t}$$

$$\text{Re } \vec{p}_{\text{total}}(\omega) = p_0 \hat{i} \cos \omega t + p_0 \hat{j} \sin \omega t$$

which is the desired answer.

Then, the " \vec{p} " to be used in the first formula above is

$$\boxed{\vec{p}_{\text{tot}} = p_0 (\hat{i} - i \hat{j})}$$

Repeating the steps done in class (and in Jackson) to get the \vec{H} of relevance in the radiation zone we get:

$$\vec{H}_{\text{tot}} = \frac{ck^2}{4\pi} (\vec{m} \times \vec{P}_{\text{tot}}) e^{\frac{ikr}{r}}$$

i.e. we simply can use the formulas previously derived just making sure that \vec{p} is \vec{p}_{tot} as defined before.

Same for $\vec{E}_{\text{tot}} = Z_0 (\vec{H}_{\text{tot}} \times \vec{m})$ in the radiation zone.

So all is the same as before, just \vec{H}_{tot} and \vec{E}_{tot} are "rotating" like the dipole $\vec{P}_{\text{tot}}(\omega)$ is doing.

Then, we can go directly to the formula for the power emitted in a particular direction, averaged over time:

$$\frac{dP}{dt} = \frac{1}{2} \frac{Z_0 c^2 k^4}{(4\pi)^2} \left| (\vec{m} \times \vec{P}_{\text{tot}}) \times \vec{m} \right|^2$$

this symbol means the dot product between a vector and its conjugate as, e.g.: $|\vec{W}|^2 = \vec{W} \cdot \vec{W}^*$

Then, we have to calculate

$$\vec{n} = (\vec{n} \times \vec{p}_{\text{tot}}) \times \vec{n}$$

and then $\vec{n} \cdot \vec{n}^*$.

It is better to work in spherical coordinates since $\vec{n} = \hat{e}_r$ has a simple expression. With regards to \vec{p}_{tot} we need to express it in spherical coordinates:

$$\vec{p}_{\text{tot}} = p_0 (\hat{i} - i \hat{j})$$

$$\hat{i} = \sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi$$

$$\hat{j} = \sin \theta \sin \phi \hat{e}_r + \cos \theta \sin \phi \hat{e}_\theta + \cos \phi \hat{e}_\phi$$

Then: $\vec{p}_{\text{tot}} =$

$$= p_0 [\sin \theta (\cos \phi - i \sin \phi) \hat{e}_r + \cos \theta (\cos \phi - i \sin \phi) \hat{e}_\theta + (-\sin \phi - i \cos \phi) \hat{e}_\phi]$$

$$= p_0 \sin \theta e^{-i\phi} \hat{e}_r + p_0 \cos \theta e^{-i\phi} \hat{e}_\theta + \underbrace{p_0 (-i)(\cos \phi - i \sin \phi)}_{e^{-i\phi}} \hat{e}_\phi$$

$$= p_r \hat{e}_r + p_\theta \hat{e}_\theta + p_\phi \hat{e}_\phi \text{ with}$$

$$p_r = p_0 \sin \theta e^{-i\phi}$$

$$p_\theta = p_0 \cos \theta e^{-i\phi}$$

$$p_\phi = p_0 (-i) e^{-i\phi}$$

$$\vec{n} \times \vec{p}_{\text{tot}} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 1 & 0 & 0 \\ p_r & p_\theta & p_\phi \end{vmatrix} = -p_\phi \hat{e}_\theta + p_\theta \hat{e}_\phi$$

$$(\vec{n} \times \vec{p}_{\text{tot}}) \times \vec{n} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 0 & -p_\phi & p_\theta \\ 1 & 0 & 0 \end{vmatrix} = +p_\theta \hat{e}_\theta + p_\phi \hat{e}_\phi$$

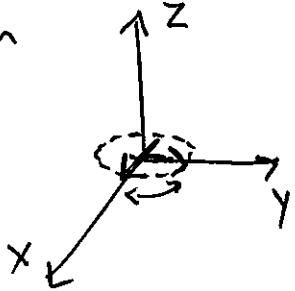
$$(\vec{n} \times \vec{p}_{\text{tot}}^*) \times \vec{n} = p_\theta^* \hat{e}_\theta + p_\phi^* \hat{e}_\phi$$

$$|(\vec{n} \times \vec{p}_{\text{tot}}) \times \vec{n}|^2 = |p_\theta|^2 + |p_\phi|^2 =$$

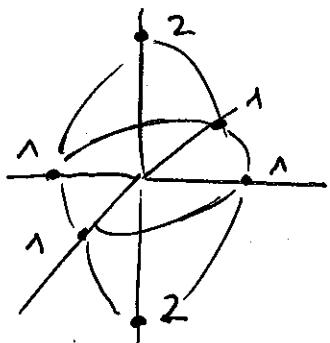
$$= p_0^2 (\cos^2 \theta + 1) . \quad \text{Then:}$$

$$\boxed{\frac{dP}{dr} = \frac{1}{2} \frac{Z_0 c^2 k^4}{(4\pi)^2} p_0^2 (1 + \cos^2 \theta)}$$

Note that the result depends only on θ . This is reasonable after the "time average" procedure since all angles in absolute enter with the same weight, making the problem invariant under rotations around the Z axis.

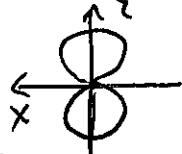


The result for the θ dependence $(1 + \cos^2 \theta)$ is (just a sketch) :



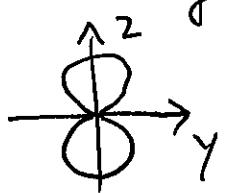
namely it is double in intensity along the z axis (i.e. $\theta = 0$ and π) than in plane (i.e. $\theta = \pi/2$).

This is reasonable at least intuitively since slope $P\hat{o}f$ should have a distribution



which is invariant under x -axis rotations.

By the same reasoning (-i) $P\hat{o}f$ should have a distribution



which is invariant under rotations with respect to y rotations.

Putting the two together gives double value along the z axis and it becomes invariant under z axis rotations.

With regards to the total power we simply need to integrate:

$$P = \frac{1}{2} \frac{Z_0 C^2 k^4}{(4\pi)^2} P_0^2 \int_0^{\pi} \int_0^{2\pi} d\theta \sin\theta (1 + \cos^2\theta)$$

$$= \frac{Z_0 C^2 k^4 P_0^2}{8\pi} \left[-\cos\theta \Big|_0^\pi - \frac{\cos^3\theta}{3} \Big|_0^\pi \right] =$$

$$= 2 + \frac{2}{3} = \frac{8}{3}$$

$$\boxed{P = \frac{Z_0 C^2 k^4 P_0^2}{6\pi}}$$

Note from (9.24) Jackson that the power of just $P_0 \hat{i}$ or $(-i)P_0 \hat{j}$ individually is $\frac{Z_0 C^2 k^4}{12\pi} P_0^2$. Thus, the sum is precisely the result shown above.

In general, this is not the case i.e. we cannot simply sum the powers of each dipole, but in this case it works (mainly because of the 90° rotation, as shown in the next couple of pages).

$$\vec{P}_{tot} = P_0(\hat{i} - i\alpha\hat{j}) \quad (\alpha \neq 1) \quad (\alpha \in \mathbb{R})$$

Consider for instance a combination

$$P_\theta = P_0 \cos\theta (\cos\phi - i\alpha \sin\phi)$$

as shown here, i.e still two dipoles 90° out of phase, but with different magnitudes.

$$P_\phi = (-i) P_0 (\cos\phi - i\alpha \sin\phi)$$

$$P_{tot}^2 + |P_\phi|^2 = P_0^2 \cos^2\theta (\cos\phi - i\alpha \sin\phi)(\cos\phi + i\alpha \sin\phi)$$

$$+ P_0^2 (\cos\phi - i\alpha \sin\phi)(\cos\phi + i\alpha \sin\phi)$$

$$= P_0^2 (\cos^2\theta + 1) (\cos\phi - i\alpha \sin\phi)(\cos\phi + i\alpha \sin\phi)$$

The power of $P_0 \hat{i}$ alone goes like $\frac{Z_0 C^2 k^4 P_0^2}{12\pi}$ ①

The total power of $(-i)\alpha P_0 \hat{j}$ alone goes like $\frac{Z_0 C^2 k^4 \alpha^2 P_0^2}{12\pi}$ ②

The sum of ① and ② gives

$$Z_0 C^2 k^4 P_0^2 \left(\frac{1+\alpha^2}{12\pi} \right) \quad (A)$$

The true total power goes like

$$\int_0^{2\pi} \int_0^{\pi} d\phi \sin\theta \underbrace{P_0^2 (\cos^2\theta + 1) (\cos\phi - i\alpha \sin\phi)(\cos\phi + i\alpha \sin\phi)}_{\cos^2\phi + \alpha^2 \sin^2\phi}$$

$$P = \frac{1}{2} \frac{Z_0 C^2 k^4 P_0^2}{4\pi} \frac{8}{3} \left(\int_0^{2\pi} \cos^2\phi d\phi + \alpha^2 \int_0^{2\pi} \sin^2\phi d\phi \right)$$

$$\int_0^{2\pi} \cos^2 \phi d\phi = \int_0^{2\pi} d\phi \left(\frac{1}{2} + \frac{\cos 2\phi}{2} \right) = \pi$$

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi = 2\cos^2 \phi - 1$$

$$\cos^2 \phi = \frac{1 + \cos 2\phi}{2}$$

$$\int_0^{2\pi} \sin^2 \phi d\phi = \int_0^{2\pi} (1 - \cos^2 \phi) d\phi = 2\pi - \pi = \pi.$$

$$P = \frac{Z_0 C^2 k^4 p_0^2}{4 \pi \times 3} \times \frac{2}{3} \pi (1 + \alpha^2) = Z_0 C^2 k^4 p_0^2 \frac{(1 + \alpha^2)}{12 \pi}$$

The same as before ⁱⁿ A !
thus introducing an "a"
is not enough to alter
the fact that just adding
individual powers is sufficient.

However, do not consider 90° rotated dipoles but
simply 2 identical ones: $\vec{P} = p_0 \hat{i} + p_0 \hat{i} = 2p_0 \hat{i}$.

Using (8.24) Jackson, then

$$P = \frac{Z_0 C^2 k^4 (2p_0)^2}{12 \pi} = 4 P_{\text{of an individual one}}$$

While just adding the individual contributions gives

$$P = 2 P_{\text{of an individual one.}}$$

Thus, if not off by 90° , the power cannot be obtained by adding the individual powers.

Said in simpler terms:

$$(1+1)^2 = 4 \neq 1^2 + 1^2 = 2$$

but $|1+i|^2 = (1+i)(1-i) = 2$ which is equal to
 $|1|^2 + |i|^2 = 2$.

or

$$|1+ai|^2 = (1+ai)(1-ai) = 1+a^2 = |1|^2 + |ai|^2$$

or more generally

$$\begin{aligned} |1+e^{i\varphi}|^2 &= (1+e^{i\varphi})(1+\bar{e}^{i\varphi}) = 1 + e^{i\varphi} + \bar{e}^{i\varphi} + 1 \\ &= 2 + 2\cos\varphi \neq |1|^2 + |e^{i\varphi}|^2 = 2; \end{aligned}$$

only if $\varphi = \pi/2$, so in our example, the power is the sum of the individual powers!