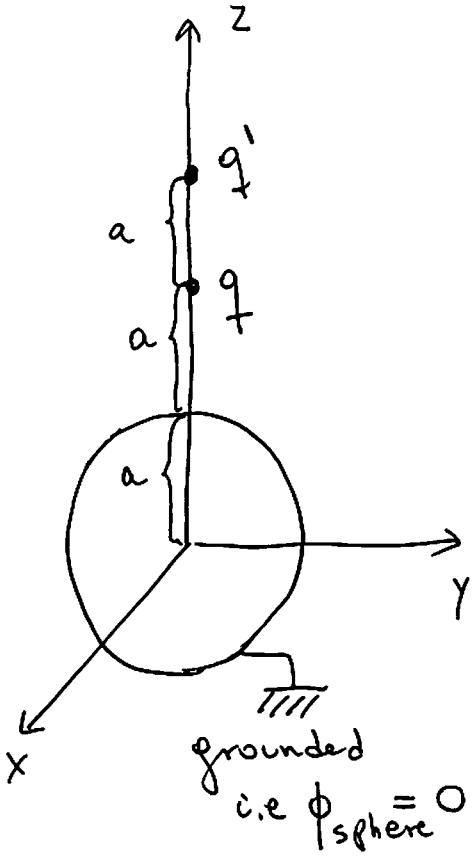
Problem 1



(a)

We can solve this problem by superposition of the solution corresponding to "q" alone and then adding the solution of "q'" alone. In both cases, $\phi_{sphere} = 0$ is used so the boundary conditions are respected.

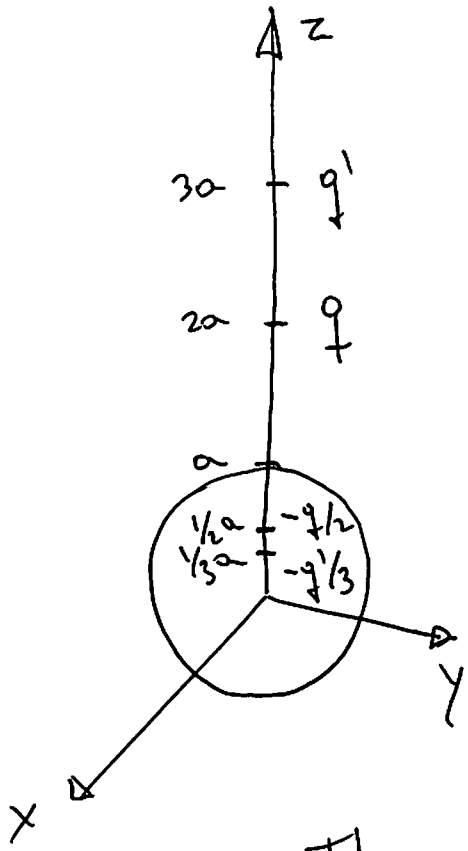
Since it is allowed to simply use information in the book about the single charge problem, we will use formulas of Section 2.2.

The image charge of q is of magnitude

$$q_{image} = -\frac{a}{2a} q = -\frac{q}{2} \quad \text{and location} \quad y_{image} = \frac{a^2}{2a} = \frac{a}{2}.$$

The image charge of q' is of magnitude

$$q'_{image} = -\frac{a}{3a} q' = -\frac{q'}{3} \quad \text{and location} \quad y'_{image} = \frac{a^2}{3a} = \frac{a}{3}.$$



In summary, the figure shows the location and magnitude of the real and image charges.

The condition of overall neutrality requires:

$$q' + q - \frac{q}{2} - \frac{q'}{3} = 0$$

Then, $q'\left(1 - \frac{1}{3}\right) = -q\left(1 - \frac{1}{2}\right)$

$$q' \frac{2}{3} = -q \frac{1}{2}$$

$$\boxed{q' = -\frac{3}{4} q}$$

(b)

In order to find the dipole moment, I will consider the sum of the potentials at large distances. The calculation is simplified if the z axis is considered.

Along the z axis the total potential is:

$$\phi(z) = \frac{1}{4\pi\epsilon_0}\left[\frac{\left(-\frac{3}{4}q\right)}{(z-3a)} + \frac{q}{(z-2a)} + \frac{\left(-\frac{q}{2}\right)}{\left(z-\frac{a}{2}\right)} + \frac{\left(\frac{q}{4}\right)}{\left(z-\frac{a}{3}\right)}\right]$$

$$\approx \frac{1}{4\pi\epsilon_0}\left[-\frac{3}{4}\frac{q}{z}\left(1+\frac{3a}{z}\right) + \frac{q}{z}\left(1+\frac{2a}{z}\right) - \frac{q}{2z}\left(1+\frac{a}{2z}\right) + \frac{q}{4z}\left(1+\frac{a}{3z}\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{z}\underbrace{\left(-\frac{3}{4}q + q - \frac{q}{2} + \frac{q}{4}\right)}_{=0} + \frac{1}{z^2}\underbrace{\left(-\frac{3}{4}q.3a + q\,2a - \frac{q}{2}\cdot\frac{a}{2} + \frac{q}{4}\cdot\frac{a}{3}\right)}_{qa\left(-\frac{9}{4}+2-\frac{1}{4}+\frac{1}{12}\right) =}\right]$$

$$= qa\,\frac{(-27+24-3+1)}{12}$$

$$= -\frac{5}{12}qa$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{z^2}\left(-\frac{5}{12}qa\right)$$

Comparing with (4.10), along the z axis, $\phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{z} + \frac{p}{z^2} + \cdots\right]$

then $q = 0$ and

$$\boxed{p = -\frac{5}{12}qa}$$

electric dipole moment. It points along the z axis in the negative direction

(c) Now let us calculate the surface charge density. From (2.5),

$$\underset{\substack{\uparrow \\ \text{induced} \\ \text{by } q}}{\sigma} = \frac{-q}{4\pi\, a \cdot 2a} \frac{\left(1 - \frac{a^2}{4a^2}\right)}{\left(1 + \frac{a^2}{4a^2} - \frac{2a}{2a}\cos\theta\right)^{3/2}} =$$

$$= \frac{-q}{8\pi a^2} \cdot \frac{3/4}{\left(\frac{5}{4} - \cos\theta\right)^{3/2}} = \frac{-3q}{32\pi a^2 \left(\frac{5}{4} - \cos\theta\right)^{3/2}}$$

$$\underset{\substack{\uparrow \\ \text{induced} \\ \text{by } q' = -\frac{3}{4}q}}{\sigma} = -\frac{\left(-\frac{3}{4}q\right)}{4\pi\, a\, 3a} \frac{\left(1 - \frac{a^2}{9a^2}\right)}{\left(1 + \frac{a^2}{9a^2} - \frac{2a}{3a}\cos\theta\right)^{3/2}} =$$

$$= \frac{q}{16\pi a^2} \cdot \frac{8/9}{\left(\frac{10}{9} - \frac{2}{3}\cos\theta\right)^{3/2}} = \frac{q}{18\pi a^2 \left(\frac{10}{9} - \frac{2}{3}\cos\theta\right)^{3/2}}$$

$$\boxed{\sigma_{total} = \frac{q}{\pi a^2}\left[-\frac{3}{32\left(\frac{5}{4} - \cos\theta\right)^{3/2}} + \frac{1}{18\left(\frac{10}{9} - \frac{2}{3}\cos\theta\right)^{3/2}}\right]}$$

This is <u>negative</u> for all θ's in the $[0, \pi]$ range.
(from "Wolfram Alpha" online)