Problem 2

(a) For $r > a$, the most general potential for this problem can be written as:

$$\phi(r,\theta) = \sum_{l} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) \qquad (r > a)$$

We know that at the surface of the sphere of radius a, the potential is $V_0 \cos\theta$ which is $V_0 P_1(\cos\theta)$. Then, the boundary condition reads:

$$V_0 P_1(\cos\theta) = \sum_{l} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$$

Matching coefficients we obtain:

$$V_0 = \frac{B_1}{a^2} \quad \text{for } l = 1$$

$$0 = B_l \quad \text{for } l \neq 1$$

potential of an electric dipole

Then, $\phi(r,\theta) = \frac{B_1}{r^2} P_1(\cos\theta) = \boxed{V_0 \frac{a^2}{r^2} \cos\theta}$

It is clear that having a $V(\theta)$ equal to a Legendre polynomial in its angular dependence establishes the same property all over space.

(b) From previous problems solved in class and from the Green function of the sphere we can write:

$$\boxed{\phi(r,\theta) = \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' V_0 \cos\theta' \frac{a\,(r^2-a^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

(c)

At first sight these two expressions seem very different! However, consider the large r limit. If in the square root in the denominator I only keep r i.e. if $(\sqrt{\ldots})^{3/2} \approx r^3$, there is a cancellation since $\int_0^{\pi} d\theta' \sin\theta' \cos\theta' = \frac{\sin^2\theta'}{2}\Big|_0^{\pi} = 0$. Then, I must keep one more term.

$$\frac{1}{(r^2+a^2-2ar\cos\gamma)^{3/2}} = \frac{1}{r^3}\frac{1}{\left(1+\frac{a^2}{r^2}-\frac{2a}{r}\cos\gamma\right)^{3/2}} \approx \frac{1}{r^3}\left(1+\frac{3}{2}\cdot\frac{2a}{r}\cos\gamma\right)$$

$$\phi(r,\theta) \approx \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' V_0 \frac{a\,r^2}{r^3}\left(1+\frac{3a}{r}\cos\gamma\right)$$

$$= \frac{1}{4\pi}\frac{a}{r}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' V_0 \frac{3a}{r}\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]$$

first term cancels, as explained before

Note that

$$\int_0^{2\pi} d\phi' \cos(\phi-\phi') = \int_0^{2\pi} d\phi' (\cos\phi \cos\phi' + \sin\phi \sin\phi') = 0$$

Then, thus far

$$\phi(r,\theta) \approx \frac{1}{4\pi} 3 \frac{a^2}{r^2} V_0 \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \cos\theta \cos\theta'$$

$$= \frac{3}{2} \frac{a^2}{r^2} \cos\theta V_0 \underbrace{\int_0^{\pi} d\theta' \cos^2\theta' \sin\theta'}_{-\frac{\cos^3\theta'}{3}\Big|_0^{\pi} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}} = \boxed{V_0 \frac{a^2}{r^2} \cos\theta}$$

This is exactly the same result found in (a).

Remarkably, if we keep further terms in the $\frac{1}{r}$ expansion, <u>all of them</u> must cancell. We will not prove that explicitly here.

(d) From the lesson learned in (a), in order to have a "quadrupolar" potential we need a $V(\theta) = V_0 \, P_2(\cos\theta) = V_0 \, \frac{1}{2}(3\cos^2\theta - 1)$.

With this potential only $B_2 \neq 0$ in the Legendre polynomial expansion outlined in (a).

From expressions such as (4.6) and (4.10) in the book, we know that "$l = 2$" is associated with the quadrupole moment "Q", and with a $\frac{1}{r^3}$ r-dependence.