Problem 3

(a) The potential of an electric dipole in a material of dielectric constant ϵ is given by

$$\phi_{dipole} = \frac{p\cos\theta}{4\pi\epsilon r^2} \quad \text{if } \vec{p} \text{ points along the z axis.}$$

For this problem with uniaxial symmetry the potential inside the sphere is

$$\phi_{in} = \frac{p\cos\theta}{4\pi\epsilon r^2} + \sum_l A_l r^l P_l(\cos\theta)$$

while outside it is

$$\phi_{out} = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

The boundary conditions are:

(i) $$-\frac{1}{a}\frac{\partial \phi_{in}}{\partial \theta}\bigg|_{r=a} = -\frac{1}{a}\frac{\partial \phi_{out}}{\partial \theta}\bigg|_{r=a}$$ (tangential component of $\vec{E}$ is continuous)

(ii) $$-\epsilon \frac{\partial \phi_{in}}{\partial r}\bigg|_{r=a} = -\epsilon_0 \frac{\partial \phi_{out}}{\partial r}\bigg|_{r=a}$$ (normal component of $\vec{D}$ is continuous)

(i) gives:

$$-\frac{1}{a}\left[\frac{P}{4\pi\epsilon a^2}\overbrace{\frac{\partial(\cos\theta)}{\partial\theta}}^{P_1(\cos\theta)} + \sum_l A_l a^l \frac{\partial}{\partial\theta} P_l(\cos\theta)\right] =$$

$$= -\frac{1}{a}\left[\sum_l B_l \frac{1}{a^{l+1}} \frac{\partial}{\partial\theta} P_l(\cos\theta)\right]$$

or, coefficient by coefficient, the following relation between A_l and B_l:

$l=1$ $\qquad \frac{P}{4\pi\epsilon a^2} + A_1 a = \frac{B_1}{a^2}$

$l\neq 1$ $\qquad A_l a^l = \frac{B_l}{a^{l+1}}$

(ii) gives:

$$-\epsilon\left[\frac{P\cos\theta}{4\pi\epsilon}\left(\frac{-2}{a^3}\right) + \sum_l A_l\, l\, a^{l-1} P_l(\cos\theta)\right] =$$

$$= -\epsilon_0 \sum_l B_l \frac{[-(l+1)]}{a^{l+2}} P_l(\cos\theta)$$

or

$$l=1 \qquad -\epsilon\left[\frac{-2P}{4\pi\epsilon a^3}+A_1\right]=-\epsilon_0\left(-\frac{2B_1}{a^3}\right)$$

$$l\neq 1 \qquad -\epsilon\, A_l\, l\, a^{l-1} = -\epsilon_0\, B_l\, \frac{[-(l+1)]}{a^{l+2}}$$

For the case $l\neq 1$ we get

$$A_l = \frac{B_l}{a^{2l+1}} \quad \text{and} \quad A_l = -B_l\,\frac{(l+1)}{l\,a^{2l+1}}\,\frac{\epsilon_0}{\epsilon}$$

These are the same equations as found before in the example of a dielectric sphere in a uniform electric field. The only solution is $A_l = B_l = 0$ $(l\neq 1)$.

For $l=1$ we find a nontrivial result. The equations are:

$$\frac{P}{4\pi\epsilon a^2}+A_1 a=\frac{B_1}{a^2} \quad \text{and} \quad \frac{2P}{4\pi a^3}-\epsilon A_1=\frac{2B_1\epsilon_0}{a^3}$$

↓ multiplying by $-\frac{\epsilon}{a}$:

$$-\frac{P}{4\pi a^3}-\epsilon A_1=-\frac{\epsilon B_1}{a^3}$$. Then:

$$\frac{2P}{4\pi a^3}-2\frac{B_1\epsilon_0}{a^3}=\frac{-P}{4\pi a^3}+\frac{\epsilon B_1}{a^3} \qquad \text{or}$$

$$\frac{3P}{4\pi a^3} = (\epsilon + 2\epsilon_0)\frac{B_1}{a^3}$$

$$B_1 = \frac{3P}{4\pi(\epsilon + 2\epsilon_0)}$$

$$A_1 = \frac{B_1}{a^3} - \frac{P}{4\pi\epsilon a^3} = \frac{3P}{4\pi a^3(\epsilon + 2\epsilon_0)} - \frac{P}{4\pi\epsilon a^3} =$$

$$= \frac{P}{4\pi a^3}\left(\frac{3}{\epsilon + 2\epsilon_0} - \frac{1}{\epsilon}\right) = \frac{P}{4\pi a^3}\frac{[3\epsilon - (\epsilon + 2\epsilon_0)]}{\epsilon(\epsilon + 2\epsilon_0)} =$$

$$= \frac{P}{4\pi a^3}\frac{2(\epsilon - \epsilon_0)}{\epsilon(\epsilon + 2\epsilon_0)}$$

Then the potential outside is

$$\boxed{\phi_{out} = \frac{B_1}{r^2}P_1(\cos\theta) = \frac{3P}{4\pi(\epsilon + 2\epsilon_0)}\frac{1}{r^2}\cos\theta} \quad (r \geq a)$$

and inside

$$\boxed{\phi_{in} = \frac{P\cos\theta}{4\pi\epsilon r^2} + \frac{P}{4\pi a^3}\frac{2(\epsilon - \epsilon_0)}{\epsilon(\epsilon + 2\epsilon_0)} r\cos\theta}$$

$$= \frac{P\cos\theta}{4\pi\epsilon r^2}\left[1 + 2\frac{r^3}{a^3}\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\right] \quad (r \leq a)$$