

(b) Check that potential is continuous at surface:

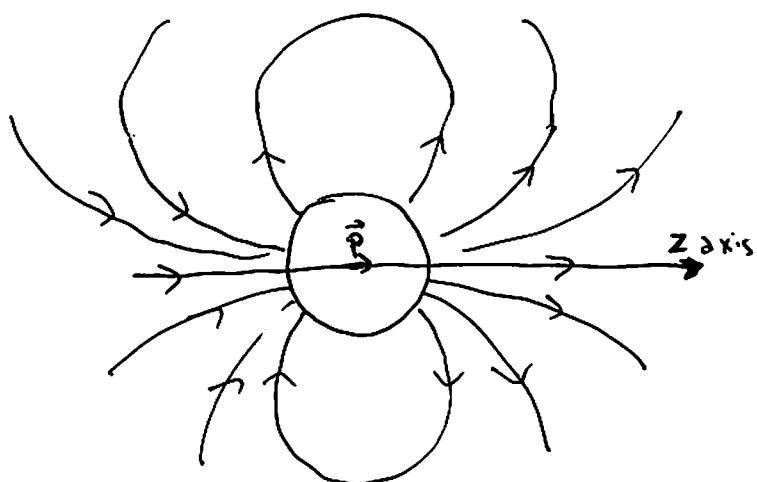
$$\phi_{\text{out}} \text{ at } r=a \text{ is } \frac{3p \cos \theta}{4\pi(\epsilon + 2\epsilon_0)a^2}$$

$$\begin{aligned} \phi_{\text{in}} \text{ at } r=a \text{ is } & \frac{p \cos \theta}{4\pi \epsilon a^2} \left[1 + 2 \frac{(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} \right] = \\ & = \frac{p \cos \theta}{4\pi \epsilon a^2} \frac{(\epsilon + 2\epsilon_0 + 2\epsilon - 2\epsilon_0)}{(\epsilon + 2\epsilon_0)} = \frac{3p \cos \theta}{4\pi a^2 (\epsilon + 2\epsilon_0)} \quad \checkmark \end{aligned}$$

Potential is indeed continuous.

At $\epsilon = \epsilon_0$, both expressions become identical
so it has to be.

(c) The potential outside is the same as a dipole
at the origin but with an effective dielectric
constant $\frac{\epsilon + 2\epsilon_0}{3}$ as opposed to ϵ_0 . Thus, the
electric field lines are as shown here.



(d) To calculate the electric field we use the expression for the gradient in spherical coordinates at the back of the book:

$$\nabla \psi = \hat{e}_r \frac{\partial \psi}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

when there is no ϕ dependence

Then, the \vec{E} inside is:

$$\vec{E}_{in} = -\nabla \phi_{in} = \hat{e}_r \left[-\frac{2}{\partial r} \left(\frac{P \cos \theta}{4\pi \epsilon r^2} + \frac{P}{4\pi a^3} \frac{2(\epsilon - \epsilon_0)}{\epsilon(\epsilon + 2\epsilon_0)} r \cos \theta \right) \right]$$

$$+ \frac{1}{r} \hat{e}_\theta \left[-\frac{2}{\partial \theta} \left(\frac{P \cos \theta}{4\pi \epsilon r^2} + \frac{P}{4\pi a^3} \frac{2(\epsilon - \epsilon_0)}{\epsilon(\epsilon + 2\epsilon_0)} r \cos \theta \right) \right] =$$

$$= -\hat{e}_r \left[\frac{P \cos \theta}{4\pi \epsilon} \left(-\frac{2}{r^3} \right) + \frac{2P(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} \cos \theta \right]$$

$$- \frac{\hat{e}_\theta}{r} \left[\frac{P}{4\pi \epsilon r^2} (-\sin \theta) + \frac{2P(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} r (-\sin \theta) \right]$$

$$\vec{P}_{in} = (\epsilon - \epsilon_0) \left\{ \hat{e}_r \left[\frac{2P \cos \theta}{4\pi \epsilon r^3} - \frac{2P(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} \cos \theta \right] + \hat{e}_\theta \left[\frac{P \sin \theta}{4\pi \epsilon r^3} + \frac{2P(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} \sin \theta \right] \right\}$$

(e)

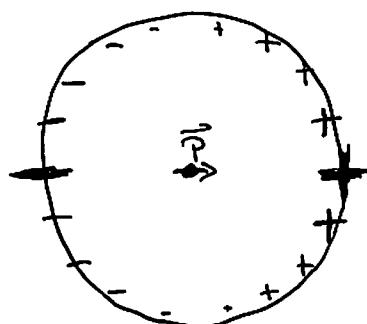
\vec{P} along the z axis inside ($\theta = 0$) is:

$$\vec{P}_{in}^{z-axis} = (\epsilon - \epsilon_0) \hat{e}_z \left[\frac{2p}{4\pi\epsilon r^3} - \frac{2p(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} \right]$$

At $r=a$:

$$\begin{aligned} P_{in} &= (\epsilon - \epsilon_0) \left[\frac{2p}{4\pi\epsilon a^3} - \frac{2p(\epsilon - \epsilon_0)}{4\pi a^3 \epsilon (\epsilon + 2\epsilon_0)} \right] = \\ &= \frac{2p(\epsilon - \epsilon_0)}{4\pi\epsilon a^3} \underbrace{\left[1 - \frac{(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} \right]}_{\frac{\epsilon + 2\epsilon_0 - \epsilon + \epsilon_0}{\epsilon + 2\epsilon_0}} = \frac{2p(\epsilon - \epsilon_0)}{4\pi\epsilon a^3} \cdot \frac{3\epsilon_0}{(\epsilon + 2\epsilon_0)} \end{aligned}$$

$$\begin{aligned} \sigma_{pol} &= -(\vec{P} - \vec{0}) \cdot \left(-\frac{\vec{r}}{r} \right) = \frac{\vec{P} \cdot \vec{r}}{r} = \vec{P} \text{ at } r=a = \\ &\stackrel{\text{at } r=a}{\uparrow} \quad \stackrel{\text{in general}}{\uparrow} \quad \stackrel{\vec{r} = \hat{e}_z}{\uparrow} \\ &\text{along the} \\ &\text{z axis} \\ &\text{in the (+) direction} & = \frac{2p(\epsilon - \epsilon_0) 3\epsilon_0}{4\pi\epsilon a^3 (\epsilon + 2\epsilon_0)} > 0 \end{aligned}$$



$$\begin{aligned} \sigma_{pol} &= -\sigma_{pol} \\ \text{at } z=-a &\parallel \text{ at } z=+a \\ \text{because now } \hat{r} &= -\hat{e}_z \end{aligned}$$

This σ_{pol} causes an internal electric field that reduces the field inside from a perfect dipole at the origin.

Outside σ_{pol} induces a dipolar electric field as found in (c).