(a) Since there are no currents, then $\vec{H} = -\nabla\phi_M$.
Furthermore, $\vec{B} = \mu\vec{H}$, and $\nabla\cdot\vec{B} = 0$.
Then, the potential satisfies the Laplace equation

$$\nabla^2\phi_M = 0$$

with the proper boundary conditions.

For $r > a$, we propose

$$\phi_M = -H_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$$

For $r < a$,

$$\phi_M = \sum_{l=0}^{\infty} \beta_l r^l P_l(\cos\theta).$$

The boundary conditions were derived in class and they are in Jackson's book.

$$\left.\frac{\partial\phi_M^{r>a}}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\phi_M^{r<a}}{\partial\theta}\right|_{r=a} ; \quad \mu_0\left.\frac{\partial\phi_M^{r>a}}{\partial r}\right|_{r=a} = \mu\left.\frac{\partial\phi_M^{r<a}}{\partial r}\right|_{r=a}$$

Remember that $\cos\theta = P_1(\cos\theta)$.

$$\phi_M^{r>a} = -H_0 r P_1(\cos\theta) + \sum \frac{\alpha_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\left.\frac{\partial \phi_M^{r>a}}{\partial \theta}\right|_{r=a} = -H_0 r \frac{\partial P_1}{\partial \theta} + \sum \frac{\alpha_\ell}{r^{\ell+1}} \left.\frac{\partial P_\ell}{\partial \theta}\right|_{r=a} =$$

$$= -H_0 a \frac{\partial P_1}{\partial \theta} + \sum \frac{\alpha_\ell}{a^{\ell+1}} \frac{\partial P_\ell}{\partial \theta}$$

$$\left.\frac{\partial \phi_M^{r<a}}{\partial \theta}\right|_{r=a} = \sum_\ell \beta_\ell a^\ell \frac{\partial P_\ell}{\partial \theta}$$

Then, for $\ell = 1$, $\boxed{-H_0 a + \frac{\alpha_1}{a^2} = \beta_1 a}$

for $\ell \neq 1$, $\boxed{\frac{\alpha_\ell}{a^{\ell+1}} = \beta_\ell a^\ell}$

The second BC gives

$$\mu_0 \left.\frac{\partial \phi_M^{r>a}}{\partial r}\right|_{r=a} = \mu_0 \left(-H_0 P_1(\cos\theta) + \sum_\ell \frac{\alpha_\ell(-\ell-1)}{a^{\ell+2}} P_\ell(\cos\theta) \right)$$

$$\mu \left.\frac{\partial \phi_M^{r<a}}{\partial r}\right|_{r=a} = \mu \left(\sum_\ell \beta_\ell \, \ell \, a^{\ell-1} P_\ell(\cos\theta) \right)$$

$$\boxed{\begin{array}{ll} \text{For } l \neq 1: & \mu_0 \dfrac{\alpha_l(-l-1)}{a^{l+2}} = \mu \beta_l l a^{l-1} \\ \text{For } l = 1: & \mu_0\left(-H_0 - \dfrac{2\alpha_1}{a^3}\right) = \mu \beta_1 \end{array}}$$

For $l \neq 1$, we cannot satisfy both equations simultaneously. Thus, $\alpha_l = \beta_l = 0$ for $l \neq 1$

For $l = 1$, and changing $\alpha_1 \to \alpha$ and $\beta_1 \to \beta$, we get:

$$-H_0 a + \frac{\alpha}{a^2} = \beta a$$

$$\left(\mu' = \frac{\mu}{\mu_0}\right)$$

$$-H_0 - \frac{2\alpha}{a^3} = \frac{\mu}{\mu_0}\beta = \mu' \beta$$

Multiplying first one by $\frac{2}{a}$ and adding to the second we get:

$$\left.\begin{array}{l} -H_0 2 + \dfrac{2\alpha}{a^3} = 2\beta \\ -H_0 - \dfrac{2\alpha}{a^3} = \mu'\beta \end{array}\right\} \to -3H_0 = (2+\mu')\beta$$

$$\boxed{\beta = \frac{-3H_0}{2+\mu'}}$$

$$\frac{\alpha}{a^2} = \beta a + H_0 a$$

$$\alpha = \beta a^3 + H_0 a^3 = \frac{-3H_0}{2+\mu'} a^3 + H_0 a^3 = H_0 a^3 \underbrace{\left(1 - \frac{3}{2+\mu'}\right)}_{\frac{2+\mu'-3}{2+\mu'} = \frac{-1+\mu'}{2+\mu'}}$$

$$\boxed{\alpha = H_0 a^3 \left(\frac{-1+\mu'}{2+\mu'}\right)}$$

Then:

$$\boxed{\begin{aligned} \phi_M^{r>a} &= -H_0 r\cos\theta + H_0 a^3\left(\frac{-1+\mu'}{2+\mu'}\right)\frac{1}{r^2}\cos\theta \\ \phi_M^{r<a} &= \frac{-3H_0}{2+\mu'}\, r\cos\theta \end{aligned}}$$

Special case $r=a$:

$$\phi_M^{r>a} = -H_0 a\cos\theta + H_0 a^3\left(\frac{-1+\mu'}{2+\mu'}\right)\frac{1}{a^2}\cos\theta$$

$$= H_0 a\cos\theta\left(-1 + \frac{(-1+\mu')}{2+\mu'}\right) = H_0 a\cos\theta\,\frac{(-2-\mu'-1+\mu')}{2+\mu'}$$

$$= \frac{-3H_0 a\cos\theta}{2+\mu'}$$

$$\phi_M^{r<a} = \frac{-3H_0 a}{2+\mu'}\cos\theta$$

They match showing that ϕ_M is continuous as it should.

(b) Let us calculate $\vec{H} = -\nabla \phi_M$ along the z-axis, where $-\nabla \equiv -\frac{d}{dz}$.

$$\phi_M^{z>a} = -H_o z + H_o a^3 \left(\frac{-1+\mu'}{2+\mu'}\right)\frac{1}{z^2}$$

$$\phi_M^{z<a} = \frac{-3H_o}{2+\mu'} z$$

Then,

$$\boxed{H^{z>a} = -\frac{d}{dz}\phi_M^{z>a} = H_o + H_o a^3 \left(\frac{-1+\mu'}{2+\mu'}\right)\frac{2}{z^3}}$$

$$\boxed{H^{z<a} = -\frac{d}{dz}\phi_M^{z<a} = +\frac{3H_o}{2+\mu'}}$$ (constant inside sphere)

Get $\vec{B}$, from $\vec{B} = \mu \vec{H}$ — μ_o outside, μ inside

$$\boxed{\vec{B}^{z>a}_{\text{along } z} = \mu_o H_o + \mu_o H_o a^3 \left(\frac{-1+\mu'}{2+\mu'}\right)\frac{2}{z^3}}$$

$$\boxed{\vec{B}^{z<a}_{\text{along } z} = +\frac{3\mu H_o}{2+\mu'}}$$

How about $\vec{M}$ along the z axis?

$$\vec{B} = \underset{\uparrow}{\mu_0}(\vec{H}+\vec{M})$$

always μ_0 in this formula.

For $z > a$, $\vec{M} = 0$ obviously

For $z < a$,

$$\vec{M}\Big|_{\text{along } z} = \frac{\vec{B}}{\mu_0} - \vec{H}\Big|_{\text{along } z} = \frac{+3\mu H_0}{2+\mu'} \cdot \frac{1}{\mu_0} - \left(+\frac{3H_0}{2+\mu'}\right) =$$

$$= \frac{3H_0}{2+\mu'}\left(+\mu' - 1\right) = \boxed{+3H_0 \frac{(-1+\mu')}{(2+\mu')}} \quad \left(\text{constant inside sphere}\right)$$

(c)

Consider now the limit $\mu \to \infty$ (i.e. $\mu' \to \infty$ as well)

$$\frac{-1+\mu'}{2+\mu'} \xrightarrow[\mu' \to \infty]{} 1, \quad \frac{1}{2+\mu'} \xrightarrow[\mu' \to \infty]{} 0$$

$$\boxed{\begin{aligned} \phi_M^{r>a} &= -H_0 r\cos\theta + H_0 \frac{a^3 \cos\theta}{r^2} \\ \phi_M^{r<a} &= 0 \end{aligned}} \quad (= 0 \text{ if } r = a)$$

$$\boxed{\begin{aligned} H^{z>a} &= H_0 + H_0 a^3 \frac{2}{z^3} \\ H^{z<a} &= 0 \end{aligned}}$$

discontinuous at $z=a$

$$\boxed{\begin{aligned} B^{z>a} &= \mu_0 H_0 + \mu_0 H_0 a^3 \frac{2}{z^3} \\ B^{z<a} &= +\frac{3\mu}{2+\frac{\mu}{\mu_0}} H_0 = +\frac{3\mu_0\mu}{2\mu_0+\mu} H_0 \longrightarrow +3\mu_0 H_0 \end{aligned}}$$

$$B^{z>a}\Big|_a = 3\mu_0 H_0$$

$$B^{z<a}\Big|_a = 3\mu_0 H_0$$

$\vec{B}$ is continuous as expected since it has no sources

The source of $\vec{H}$ at the surface is $\nabla\cdot\vec{M}$ since $\vec{M}$ changes abruptly at the surface.

(d)

It is quite interesting that the $\vec{B}$ field inside along the z axis is larger than outside. The lines of $\vec{B}$ field tend to accumulate in a region of large μ, thus explaining this effect.