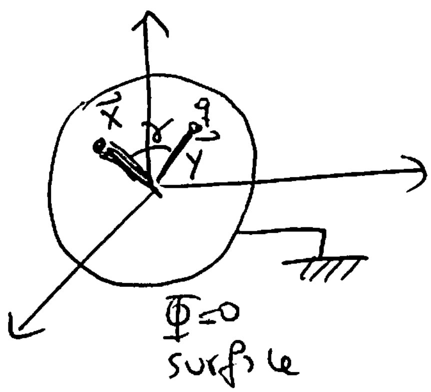
(a) From the solution of problem 2.2 (a)

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+y^2-2ry\cos\gamma}} - \frac{(a/y)}{\sqrt{r^2+\frac{a^4}{y^2}-2r\frac{a^2}{y}\cos\gamma}}\right]$$




$\vec{x}$ is the observation point
$\vec{y}$ is position of charge.

Construction of $G_D(\vec{x},\vec{x}')$ requires the identification of $\vec{x}'$ with the position of the charge $q$, and also $q/4\pi\epsilon_0 = 1$ (all this was discussed in class)

Then, simply:

$$G_D(\vec{x},\vec{x}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{(a/r')}{\sqrt{r^2+\frac{a^4}{r'^2}-2r\frac{a^2}{r'}\cos\gamma}}$$

$r=|\vec{x}|$
$r'=|\vec{x}'|$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

It is Dirichlet because $\Phi=0$ at surface in Problem 2.2(a).

(b) To construct the integral expression we need

$$\left.\frac{\partial G_D}{\partial n'}\right|_{\text{surface}} = \vec{n}' \cdot \nabla' G_D \Big|_{\text{surface}}.$$

$\vec{n}'$ points to the outside (because the volume of interest is the inside), and in the radial direction. Then $\vec{n}' = \hat{e}_r$, and only the $\hat{e}_r$ component of $\nabla' G_D$ is needed.

$$\vec{n}' \cdot \nabla' G_D = \frac{\partial}{\partial r'} G_D$$

$$\frac{\partial}{\partial r'}\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{(a/r')}{\sqrt{r^2+\frac{a^4}{r'^2}-\frac{2ra^2\cos\gamma}{r'}}}\right]_{r'=a} =$$

numentor goes inside $\Gamma$.

$$= \left(-\frac{1}{2}\right)\frac{(2r'-2r\cos\gamma)}{(r^2+r'^2-2rr'\cos\gamma)^{3/2}} - \left(-\frac{1}{2}\right)\frac{\left(\frac{2r^2r'}{a^2}-2r\cos\gamma\right)}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{3/2}}\Bigg]_{r'=a}$$

$$= \left(-\frac{1}{2}\right)\frac{(2a-2r\cos\gamma)}{(a^2+r^2-2ra\cos\gamma)^{3/2}} + \frac{1}{2}\,\frac{\left(\frac{2r^2}{a}-2r\cos\gamma\right)}{(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

$$= \frac{\left(-a+\frac{r^2}{a}\right)}{(r^2+a^2-2ra\cos\gamma)^{3/2}}$$

still depends on $\theta'$ and $\phi'$

Finally:

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da' =$$

$$= \overbrace{\underbrace{-\frac{a^2}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\,d\theta'\;\underbrace{\Phi(a,\theta',\phi')}_{\text{given}}\;\frac{\left(-a+\frac{r^2}{a}\right)}{\left(r^2+a^2-2ra\cos\gamma\right)^{3/2}}}}$$

$\cos\gamma$ depends on $\theta'$ and $\phi'$