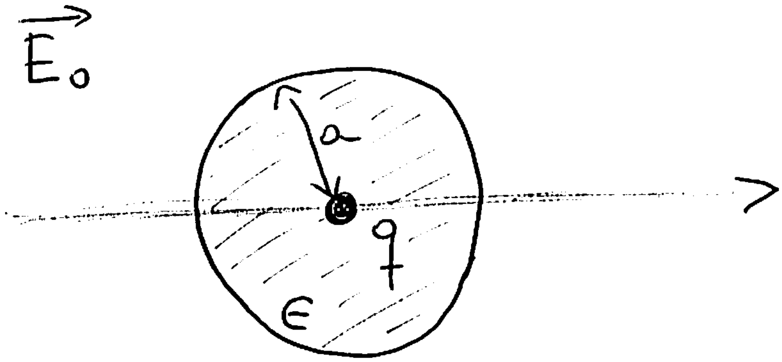
(a)



$$\phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \frac{q}{4\pi\epsilon r}$$

$$\phi_{out} = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) - E_0 r\cos\theta$$

Note that $-E_0 r\cos\theta = -E_0 r \overbrace{P_1(\cos\theta)}^{\cos\theta}$

and $\frac{q}{4\pi\epsilon r} = \frac{q}{4\pi\epsilon r} \underbrace{P_0(\cos\theta)}_{1}$

Then, it is expected that $l=0$ and 1 can be special. Let us rewrite the potentials as:

$$\phi_{in} = A_0 + A_1 r\cos\theta + \sum_{l=2}^{\infty} A_l r^l P_l(\cos\theta) + \frac{q}{4\pi\epsilon r}$$

$$\phi_{out} = \frac{B_0}{r} + \frac{B_1}{r^2} P_1(\cos\theta) + \sum_{l=2}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) - E_0 r\cos\theta$$

Requesting $\phi_{in} = \phi_{out}$ at $r = a$ and comparing coefficients we get:

$$\boxed{\begin{aligned} A_0 + \frac{q}{4\pi\epsilon a} &= \frac{B_0}{a} \quad ; \quad l = 0 \\ A_1 a &= \frac{B_1}{a^2} - E_0 a \quad ; \quad l = 1 \\ A_l a^l &= \frac{B_l}{a^{l+1}} \quad ; \quad l = 2, 3, \dots \end{aligned}}$$

From " $(\vec{D}_2 - \vec{D}_1) \cdot \vec{n}_{21} = 0$ " we get:

$$\epsilon \left. \frac{\partial \phi_{in}}{\partial r} \right|_{r=a} = \epsilon_0 \left. \frac{\partial \phi_{out}}{\partial r} \right|_{r=a}$$

$$\epsilon \left[A_1 \cos\theta + \sum_{l=2}^{\infty} A_l \, l \, a^{l-1} P_l(\cos\theta) - \frac{q}{4\pi\epsilon a^2} \right] =$$

$$= \epsilon_0 \left[-\frac{B_0}{a^2} - \frac{2B_1}{a^3} \cos\theta + \sum_{l=2}^{\infty} B_l \frac{(-l-1)}{a^{l+2}} P_l(\cos\theta) - E_0 \cos\theta \right]$$

$$\boxed{\begin{array}{lll} -\epsilon \dfrac{q}{4\pi\epsilon a^2} = -\dfrac{\epsilon_0 B_0}{a^2} & ; & l=0 \\ \epsilon A_1 = -2\dfrac{\epsilon_0 B_1}{a^3} - \epsilon_0 E_0 & & l=1 \\ \epsilon A_l\, l\, a^{l-1} = \dfrac{-(l+1) B_l}{a^{l+2}} \epsilon_0 & ; & l=2,3,\dots \end{array}}$$

First consider $l=2,3,\dots$. The equations are

$$A_l a^l = \frac{B_l}{a^{l+1}} \quad ; \quad \epsilon A_l\, l\, a^{l-1} = -\frac{(l+1)}{a^{l+2}} B_l \epsilon_0$$

The only solution is $\boxed{A_l = B_l = 0.}$

Now $l=0$:

$$A_0 + \frac{q}{4\pi\epsilon a} = \frac{B_0}{a} \quad ; \quad -\frac{q}{4\pi a^2} = -\frac{\epsilon_0 B_0}{a^2}$$

$$\boxed{B_0 = \frac{q}{4\pi\epsilon_0} \quad ; \quad A_0 = \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon a}.}$$

Now $l=1$:

$$A_1 a = \frac{B_1}{a^2} - E_0 a$$

$$\epsilon A_1 = -\frac{2\epsilon_0}{a^3} B_1 - \epsilon_0 E_0$$

$$A_1 = \frac{B_1}{a^3} - E_0 = -\frac{2\epsilon_0}{a^3 \epsilon} B_1 - \frac{\epsilon_0}{\epsilon} E_0$$

$$\frac{B_1}{a^3}\left(1 + \frac{2\epsilon_0}{\epsilon}\right) = E_0 \frac{(\epsilon - \epsilon_0)}{\epsilon}$$

$$\boxed{B_1 = \frac{E_0 a^3 (\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)}}$$

$$\boxed{A_1 = \frac{E_0(\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} - E_0 = \frac{E_0(\epsilon - \epsilon_0 - \epsilon - 2\epsilon_0)}{\epsilon + 2\epsilon_0} = \frac{-3\epsilon_0 E_0}{\epsilon + 2\epsilon_0}}$$

Putting all together:

$$\boxed{\begin{aligned} \phi_{in} &= \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon a} - \frac{3\epsilon_0 E_0}{(\epsilon + 2\epsilon_0)} r\cos\theta + \frac{q}{4\pi\epsilon r}; \\ \phi_{out} &= \frac{q}{4\pi\epsilon_0 r} + \frac{E_0 a^3 (\epsilon - \epsilon_0)}{(\epsilon + 2\epsilon_0)} \frac{\cos\theta}{r^2} - E_0 r\cos\theta. \end{aligned}}$$

From book

(b) $\nabla\psi = \hat{e}_1 \frac{\partial\psi}{\partial r} + \hat{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ (if no ϕ dependence)

$$\boxed{\begin{aligned}\vec{E}_{in} = -\nabla\phi_{in} &= +\frac{q}{4\pi\epsilon r^2}\hat{e}_r + \frac{3\epsilon_0 E_0 \cos\theta}{(\epsilon+2\epsilon_0)}\hat{e}_r \\ &+ \hat{e}_\theta \frac{3\epsilon_0 E_0}{(\epsilon+2\epsilon_0)}(-\sin\theta).\end{aligned}}$$

Superposition of point-like charge and constant electric field along z axis.

$$\boxed{\begin{aligned}\vec{E}_{out} = -\nabla\phi_{out} &= \frac{q}{4\pi\epsilon_0 r^2}\hat{e}_r + E_0\cos\theta\,\hat{e}_r \\ &\frac{E_0 a^3(\epsilon-\epsilon_0)}{(\epsilon+2\epsilon_0)}\cos\theta\,\frac{2}{r^3}\hat{e}_r - E_0\sin\theta\,\hat{e}_\theta \\ &+ \frac{E_0 a^3(\epsilon-\epsilon_0)}{(\epsilon+2\epsilon_0)}\frac{1}{r^3}\sin\theta\,\hat{e}_\theta.\end{aligned}}$$

(i) It is interesting that outside the point-like electric field has $\frac{1}{4\pi\epsilon_0}$ which is correct.

(ii) The results are merely a superposition of $l=0$ and $l=1$, where $l=1$ is identical to all results obtained before in the lecture.

Consider $E_0 = 0$. Then

(c)

$$\phi_{in} = \frac{q}{4\pi\epsilon_0 a} - \frac{q}{4\pi\epsilon a} + \frac{q}{4\pi\epsilon r}$$

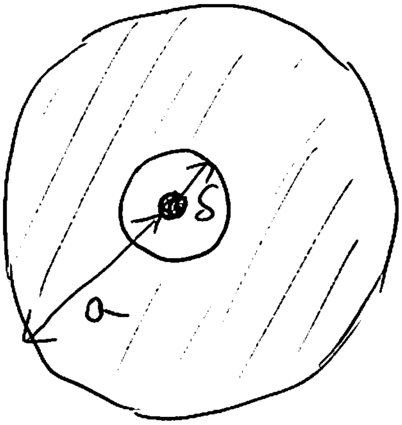
$$\phi_{out} = \frac{q}{4\pi\epsilon_0 r}$$

$$\vec{E}_{in} = \frac{q}{4\pi\epsilon r^2}\hat{e}_r \; ; \; \vec{E}_{out} = \frac{q}{4\pi\epsilon_0 r^2}\hat{e}_r .$$

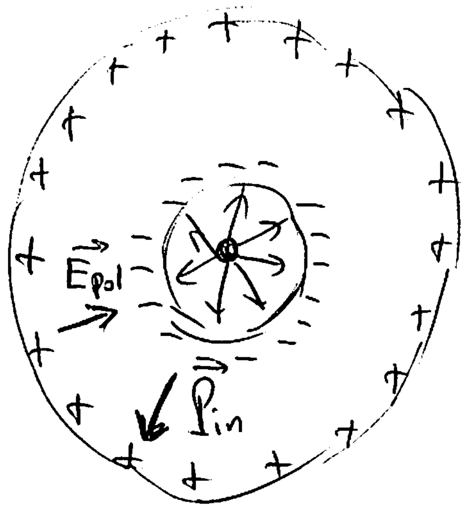
$$\boxed{\vec{P}_{in} = (\epsilon - \epsilon_0) . \underbrace{\frac{q}{4\pi\epsilon r^2}\hat{e}_r}_{\vec{E}_{in}}} \qquad \vec{P}_{out} = 0$$

$$\begin{aligned} \vec{D}_{in} &= \epsilon_0 \vec{E}_{in} + \vec{P}_{in} \\ &= \epsilon_0 \vec{E}_{in} + (\epsilon - \epsilon_0)\vec{E}_{in} \\ &= \epsilon \vec{E}_{in} \end{aligned}$$

(d) To understand intuitively the direction of $\vec{P}_{in}$ imagine the charge q inside a hollow sphere of radius δ, with δ small:



In this case the electric field $\frac{q}{4\pi\epsilon_0}\hat{e}_r$ at the center will cause a polarization with the signs as indicated:



The polarization points from "−" to "+" thus it points away from the center as in the mathematical formula $\vec{P}_{in}$.

The electric field $\vec{E}_{pol}$ goes from (+) to (−) and reduces the electric field $\frac{q}{4\pi\epsilon_0}\hat{e}_r$ leading to $\frac{q}{4\pi\epsilon}\hat{e}_r$.

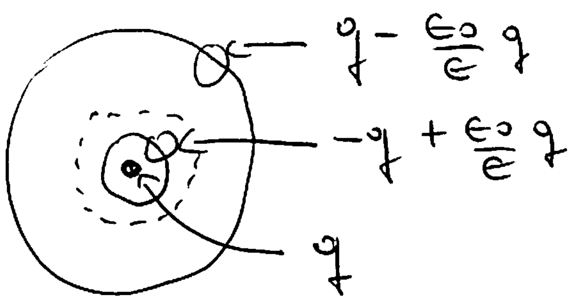
(e) $\sigma_{pol} = -(\underbrace{\vec{P}_{out}}_{=0} - \vec{P}_{in}) \cdot \underbrace{\vec{n}_{12}}_{\hat{e}_r}$

$$= (\epsilon - \epsilon_0)\frac{q}{4\pi\epsilon a^2}\underbrace{\hat{e}_r \cdot \hat{e}_r}_{1} = \underbrace{\frac{(\epsilon-\epsilon_0)\, q}{4\pi\epsilon a^2}}_{\text{independent of } \theta.}$$

Then, $q_{pol} = \frac{(\epsilon-\epsilon_0)\, q}{4\pi\epsilon a^2} \cdot \underbrace{4\pi a^2}_{Area} = \left(\frac{\epsilon-\epsilon_0}{\epsilon}\right) q$

$$= q - \frac{\epsilon_0}{\epsilon} q.$$

It seems paradoxical that there is, naively, a charge q at the center and adding to q_{pol} we get $2q - \frac{\epsilon_0}{\epsilon} q$ total, which cannot be. But the trick of the previous item of the small sphere of radius δ explains the paradox. In this context we get a $q_{pol} = -q + \frac{\epsilon_0}{\epsilon} q$ at the surface of radius δ (the sphere must be neutral)



Total is q, which is correct.

The total in the ◌ region is $\frac{\epsilon_0}{\epsilon} q$ which is the <u>effective</u> charge in the medium because

$$\frac{1}{4\pi\epsilon}\frac{q}{r} = \frac{1}{4\pi\epsilon_0}\left(\frac{\epsilon_0}{\epsilon} q\right)\frac{1}{r} .$$