

(2)

Problem 1

From Eq.(3.33) Jackson, the general potential for a problem with azimuthal symmetry is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \quad (\text{dropping the } r^l \text{ term that diverges at large } r)$$

for large r .

We also know that $P_0(\cos \theta) = \cos \theta$ (from Eq(3.15)). Then, asking that $\Phi(r, \theta)$ matches the value given at $r=a$, we get:

$$V_0 \cos \theta = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos \theta)$$

which is satisfied if $B_l = 0$ ($l \neq 1$)

$$B_1 = a^2 V_0 \text{ if } l=1$$

Then,

$$\Phi(r, \theta) = \frac{a^2 V_0}{r^2} \cos \theta.$$

From Eq.(4.10) we know that the potential of a dipole is

$$\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \vec{x}}{r^3}$$

If we orient the dipole along the z -axis,

$$\text{then } \vec{P} \cdot \vec{x} = P \hat{e}_z \cdot \vec{x} = P r \cos \theta$$

$$\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{p \cos\theta}{r^2} = \frac{\alpha^2 V_0 \cos\theta}{r^2} = \Phi(r, \theta) \quad r > a$$

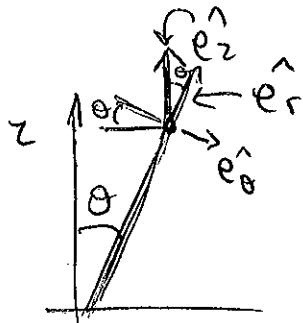
Then, $P = 4\pi\epsilon_0 \alpha^2 V_0$

(b) Now we can use Eq. (9.19)

$$\vec{H} = \frac{ck^2}{4\pi} \frac{e^{ikr}}{r} (\vec{n} \times \vec{p}) \quad (k = \frac{\omega}{c})$$

$$\vec{n} = \hat{e}_r$$

$$\vec{p} = p \hat{e}_z = p(\cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta)$$



$$\vec{n} \times \vec{p} = p \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = -p \sin\theta \hat{e}_\phi$$

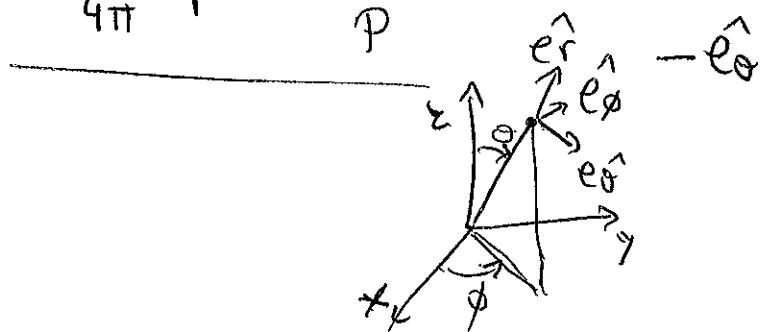
Then,

$$\vec{H} = \frac{ck^2}{4\pi} \cdot \frac{e^{ikr}}{r} \underbrace{\frac{4\pi\epsilon_0 \alpha^2 V_0}{P} \sin\theta}_{\Phi} (\hat{e}_\phi)$$

From (9.19):

$$\vec{E} = Z_0 (\vec{H} \times \vec{n}) = Z_0 \frac{ck^2}{4\pi} \frac{e^{ikr}}{r} \underbrace{4\pi\epsilon_0 a^2 V_0 \sin\theta}_{P} \underbrace{(-\hat{e}_\phi \times \hat{e}_r)}_{-\hat{e}_\theta}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$



(c) We need to calculate $\vec{n} \cdot (\vec{E} \times \vec{H}^*)$

$$\vec{E} \times \vec{H}^* = Z_0 \left(\frac{ck^2}{4\pi} \frac{P \sin\theta}{r} \right)^2 \underbrace{(\hat{e}_\theta \times \hat{e}_\phi)}_{\hat{e}_r}$$

$$\vec{n} \cdot (\vec{E} \times \vec{H}^*) = Z_0 \left(\frac{ck^2}{4\pi} \frac{P \sin\theta}{r} \right)^2$$

\hat{e}_r

$$\frac{dP}{dr} = \frac{1}{2} r^2 [\vec{n} \cdot (\vec{E} \times \vec{H}^*)] = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{ck^2}{4\pi} P \sin\theta \right)^2$$

$$\frac{dP}{dr} = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{c \omega^2}{4\pi C^2} P \sin\theta \right)^2 = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\omega^4}{(4\pi)^2 C^2} \left(4\pi \epsilon_0 a^2 V_0 \right)^2 \sin^2\theta$$

$$\text{But } \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\epsilon_0^2}{C^2} = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\epsilon_0^2}{C^3} C = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\epsilon_0^2}{C^2} \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{\epsilon_0}{C^3}$$

$$) \boxed{\frac{dP}{d\Omega} = \frac{1}{2} \frac{\omega^4}{(4\pi)^2} \frac{\epsilon_0}{C^3} (4\pi)^2 a^4 V_0^2 \sin^2 \theta}$$

$$= \frac{1}{2} \frac{\epsilon_0}{C^3} V_0^2 a^4 \omega^4 \sin^2 \theta$$

(d)

$$P = \int d\Omega \left(\frac{dP}{d\Omega} \right) =$$

$$= \frac{1}{2} \frac{\epsilon_0}{C^3} V_0^2 a^4 \omega^4 \left\{ \int_0^{2\pi} d\phi \left\{ \int_0^{\pi} d\theta \sin \theta \sin^3 \theta \right. \right. \\ \left. \left. - \int_0^{\pi} d\theta \sin \theta (1 - \cos^2 \theta) \right\} \right\} =$$

$$= \int_0^{\pi} d\theta \sin \theta - \int_0^{\pi} d\theta \sin \theta \cos^2 \theta \\ - \cos \theta \Big|_0^{\pi} - \frac{\cos^3 \theta}{3} \Big|_0^{\pi}$$

$$= 2 + \frac{1}{3}(-2) = \frac{4}{3}$$

$$\boxed{P = \frac{4}{3} \pi \frac{\epsilon_0}{C^3} V_0^2 a^4 \omega^4}$$