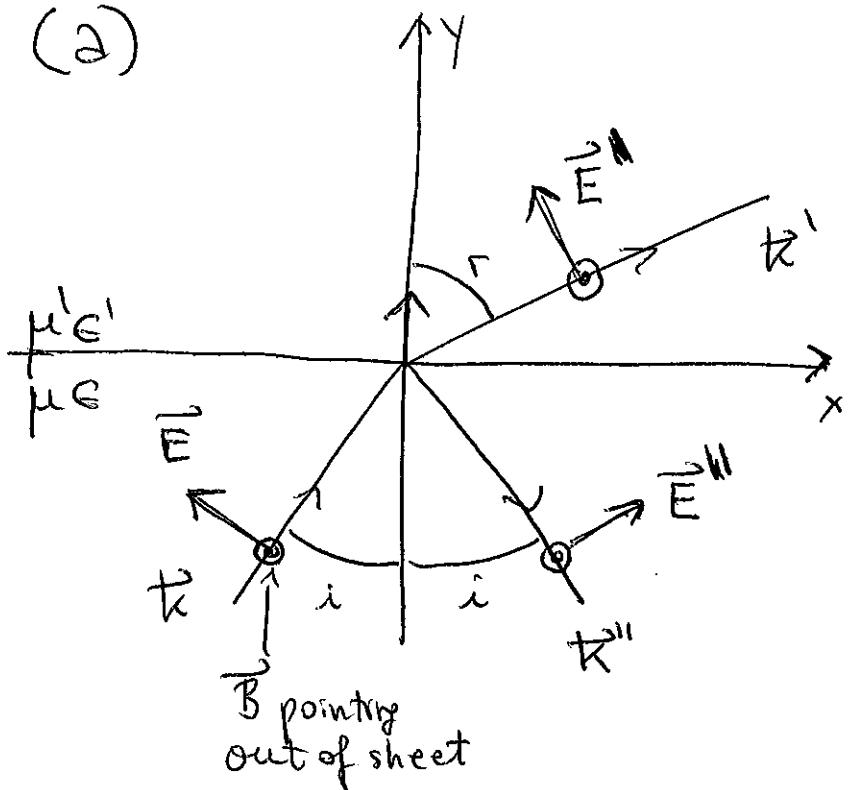


Final
Problem 1, Exam Spring 2012

) (a)



This is the figure describing the arrangement of fields for the problem we need to solve, according to Figure 7.6(b)

)

Let us consider the first boundary condition equation, that in generic terms is

$$[\epsilon(\vec{E}_0 + \vec{E}_0'') - \epsilon' \vec{E}'_0] \cdot \vec{n} = 0 \quad (1)$$

Here $\vec{n} = (0, 1, 0)$ according to the convention for the axes in the figure.

$$\vec{E}_0 = E_0 (-\cos(i), \sin(i), 0)$$

$$\vec{E}_0'' = E_0'' (+\cos(i), \sin(i), 0)$$

$$\vec{E}'_0 = E'_0 (-\cos(r), \sin(r), 0)$$

Then, $\vec{E}_0 \cdot \vec{n} = E_0 \sin(i)$

) $\vec{E}'_0 \cdot \vec{n} = E'_0 \sin(r)$

$\vec{E}''_0 \cdot \vec{n} = E''_0 \sin(i)$

and the boundary condition equation becomes:

$$\epsilon (E_0 \sin(i) + E''_0 \sin(i)) - \epsilon' E'_0 \sin(r) = 0$$

$$\epsilon(E_0 + E''_0) \sin(i) = \epsilon' E'_0 \sin(r)$$

Consider now the second equation for the boundary conditions i.e.

) $(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}''_0 - \vec{k}' \times \vec{E}'_0) \cdot \vec{n} = 0. \quad (2)$

Since both \vec{k} and \vec{E}_0 are in the (xy) plane, then its cross product points in the z direction, and then the dot product with \vec{n} gives zero.

Thus, since the same happens for $(\vec{k}'' \times \vec{E}''_0) \cdot \vec{n}$ and $(\vec{k}' \times \vec{E}'_0) \cdot \vec{n}$, the second equation gives nothing.

Consider now the third boundary condition equation i.e.

) $(\vec{E}_0 + \vec{E}''_0 - \vec{E}'_0) \times \vec{n} = 0 \quad (3)$

$$\vec{E}_0 \times \vec{n} = E_0 \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 1 & 0 \end{vmatrix} = E_0 [-\cos(\alpha)] \hat{e}_z$$

$$\vec{E}_0'' \times \vec{n} = E_0'' \cos(\alpha) \hat{e}_z$$

$$\vec{E}_0' \times \vec{n} = E_0' [-\cos(\alpha)] \hat{e}_z$$

Since all point in the same direction then equation ③ gives:

$$-E_0 \cos(\alpha) + E_0'' \cos(\alpha) - E_0' [-\cos(\alpha)] = 0$$

$$-(E_0 - E_0'') \cos(\alpha) + E_0' \cos(\alpha) = 0$$

$$(E_0 - E_0'') \cos(\alpha) = E_0' \cos(\alpha)$$

The fourth and last boundary condition equation is

$$\left[\frac{\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0''}{\mu} - \frac{\vec{k}' \times \vec{E}_0'}{\mu'} \right] \times \vec{n} = 0 \quad (4)$$

$$(\vec{k} \times \vec{E}_0) \times \vec{n} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ k \sin(i) & k \cos(i) & 0 \\ -E_0 \cos(i) & E_0 \sin(i) & 0 \end{vmatrix} \times \vec{n} =$$

$$= (0, 0, k E_0 \underbrace{[\sin^2(i) + \cos^2(i)]}_1) \times \vec{n}$$

$$= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & k E_0 \\ 0 & 1 & 0 \end{vmatrix} = -k E_0 \hat{e}_x$$

$$(\vec{k}'' \times \vec{E}_0'') \times \vec{n} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ k'' \sin(i) & -k'' \cos(i) & 0 \\ E_0'' \cos(i) & E_0'' \sin(i) & 0 \end{vmatrix} \times \vec{n} =$$

$$= (0, 0, k'' E_0'' \underbrace{[\sin^2(i) + \cos^2(i)]}_1) \times \vec{n}$$

$$= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & k E_0'' \\ 0 & 1 & 0 \end{vmatrix} = -k E_0'' \hat{e}_x$$

like $(\vec{k} \times \vec{E}_0) \times \vec{n}$ replacing unprimed by primed

$$(\vec{k}' \times \vec{E}_0') \times \vec{n} = -k' E_0' \hat{e}_x$$

Then, the fourth equation becomes:

$$-\frac{k(E_0 + E_0'')}{\mu} - \frac{(-k'E_0')}{\mu'} = 0$$

$$\boxed{k \frac{(E_0 + E_0'')}{\mu} = \frac{k'E_0'}{\mu'}}$$

(b)

The three ~~equations~~^{formed} then are:

$$\frac{E_0'}{E_0 + E_0''} = \frac{\epsilon \sin(i)}{\epsilon' \sin(r)} \quad (i)$$

$$\frac{E_0'}{E_0 - E_0''} = \frac{\cos(i)}{\cos(r)} \quad (ii)$$

$$\frac{E_0'}{E_0 + E_0''} = \frac{k}{\mu} \frac{\mu'}{k'} \quad (iii)$$

Note that Eq. (7.36) of the book says

$$\frac{\sin(i)}{\sin(r)} = \sqrt{\frac{\mu' \epsilon'}{\mu \epsilon}}. \text{ Thus } \underbrace{\frac{\epsilon}{\epsilon'} \frac{\sin(r)}{\sin(i)}}_{\text{in (i)}} = \sqrt{\frac{\mu'}{\mu} \frac{\epsilon}{\epsilon'}}$$

$$\text{But } \frac{k}{\mu} \frac{\mu'}{k'} = \frac{\mu'}{\mu} \sqrt{\frac{\mu' \epsilon'}{\mu' \epsilon'}} = \sqrt{\frac{\mu'}{\mu} \frac{\epsilon'}{\epsilon'}}$$

in (iii) 7.36 again

Then Eq (iii) has the same information as (i).

Using 7.36 again we then have that (ii) becomes:

$$\begin{aligned} \frac{E_0'}{E_0 - E_0''} &= \frac{\cos(i)}{\cos(r)} = \frac{\sqrt{1 - \sin^2(i)}}{\sqrt{1 - \sin^2(r)}} = \frac{\sqrt{1 - \sin^2(i)}}{\sqrt{1 - \sin^2(\alpha) \left(\frac{n}{n'}\right)^2}} \\ &= \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} \quad (\text{I}) \end{aligned}$$

Eq (iii) becomes

$$\frac{E_0'}{E_0 + E_0''} = \frac{\mu}{\mu} \frac{n}{n'} \quad (\text{II})$$

$$\text{Or } (E_0 + E_0'') \frac{\mu}{\mu} \frac{n}{n'} = (E_0 - E_0'') \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}}$$

$$E_0 \left(\frac{\mu}{\mu} \frac{n}{n'} - \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} \right) = -E_0'' \left(\frac{\mu}{\mu} \frac{n}{n'} + \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} \right)$$

$$\begin{aligned}
 \frac{E_0''}{E_0} &= \frac{\frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} - \frac{\mu' n}{\mu} \frac{n}{n'}}{\frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} + \frac{\mu' n}{\mu} \frac{n}{n'}} = \\
) E_0 &\frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} + \frac{\mu' n}{\mu} \frac{n}{n'} \\
 &= \frac{\mu(n')^2 \cos(i) - \mu' n \sqrt{n'^2 - n^2 \sin^2(i)}}{\mu(n')^2 \cos(i) + \mu' n \sqrt{n'^2 - n^2 \sin^2(i)}} \\
 &= \boxed{\frac{\frac{\mu(n')^2 \cos(i)}{\mu'} - n \sqrt{n'^2 - n^2 \sin^2(i)}}{\frac{\mu(n')^2 \cos(i)}{\mu'} + n \sqrt{n'^2 - n^2 \sin^2(i)}} = \frac{E_0''}{E_0}}
 \end{aligned}$$

) This is Eq.(7.41) of the book.

$$\text{From (I): } E_0' = (E_0 - E_0'') \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}}$$

$$\begin{aligned}
 \frac{E_0'}{E_0} &= \left(1 - \frac{E_0''}{E_0}\right) \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} \\
) &= \left[1 - \left(\frac{\frac{\mu(n')^2 \cos(i)}{\mu'} - n \sqrt{n'^2 - n^2 \sin^2(i)}}{\frac{\mu(n')^2 \cos(i)}{\mu'} + n \sqrt{n'^2 - n^2 \sin^2(i)}}\right)\right] \frac{\cos(i) n'}{\sqrt{n'^2 - n^2 \sin^2(i)}} =
 \end{aligned}$$

$$= \frac{2m \sqrt{\frac{1}{\mu'(n')^2 \cos(\alpha) + m^2}} \cdot \frac{\cos(\alpha) n'}{\sqrt{r}}}{=}$$

$$= \frac{2mn' \cos(\alpha)}{\frac{\mu(n')^2 \cos(\alpha) + m \sqrt{n'^2 - m^2 \sin^2(\alpha)}}{\mu'}} = \frac{E_0'}{E_0}$$

This is equation (7.41) of the book.

(c)

The reflected wave can vanish if the numerator in the formula for $\frac{E_0'}{E_0}$ vanishes.
that occurs when:

$$\frac{\mu(n')^2 \cos(\alpha)}{\mu'} = m \sqrt{n'^2 - m^2 \sin^2(\alpha)}$$

$$\left[\frac{\mu(n')^2}{\mu'} \right]^2 \underbrace{\cos^2(\alpha)}_{1 - \sin^2(\alpha)} = m^2 \left[n'^2 - m^2 \sin^2(\alpha) \right]$$

$$\left[\frac{\mu(n')^2}{\mu'} \right]^2 - m^2 n'^2 = -m^4 \sin^2(\alpha) + \sin^2(\alpha) \left[\frac{\mu(n')^2}{\mu'} \right]^2$$

$$\sin^2(\alpha) = \frac{\left[\frac{\mu(n')^2}{\mu'} \right]^2 - m^2 n'^2}{\left[\frac{\mu(n')^2}{\mu'} \right]^2 - m^4}$$

If $\mu = \mu'$, then:

$$) \quad \sin^2(i_B) = \frac{(n')^2 [(n')^2 - n^2]}{(n')^4 - n^4} = \frac{(n')^2 [(n')^2 - n^2]}{\cancel{[(n')^2 - n^2]} \cancel{[(n')^2 + n^2]}}$$

$$= \frac{(n')^2}{(n')^2 + n^2} = \frac{1}{1 + \left(\frac{n}{n'}\right)^2}$$

$$\cos^2(i_B) = 1 - \sin^2(i_B) = 1 - \frac{1}{1 + \left(\frac{n}{n'}\right)^2} = \frac{\left(\frac{n}{n'}\right)^2}{1 + \left(\frac{n}{n'}\right)^2}$$

$$) \quad \frac{\sin^2(i_B)}{\cos^2(i_B)} = \left(\frac{n'}{n}\right)^2 \text{ or } \boxed{\tan(i_B) = \frac{n'}{n}}$$

For a discussion see
page 307 of Jackson

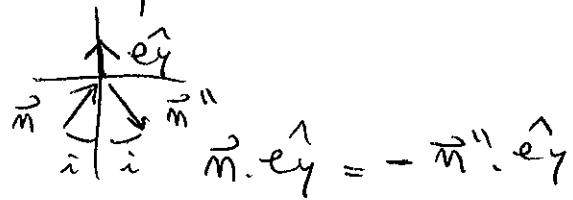
(d) To calculate the reflection and transmission coefficients we need to reason using the Poynting vector, and eq. (7.13):

$$\vec{S} = \frac{1}{2} \sqrt{\epsilon'} \frac{1}{\mu} |E_0|^2 \vec{n} \quad \text{valid for each plane wave}$$

This " \vec{n} " is in the direction of the wavevector of the wave (i.e. \vec{k})

Let us calculate the flux of energy \perp to the plane where the scattering occurs i.e. let us calculate $\vec{S} \cdot \hat{e}_y$

$$R = \left| \frac{\vec{S}_{\text{reflected}} \cdot \hat{e}_y}{\vec{S} \cdot \hat{e}_y \text{ incident}} \right| = \left| \frac{\frac{1}{2} \sqrt{\epsilon'} |E_0''|^2 (\vec{n}'' \cdot \hat{e}_y)}{\frac{1}{2} \sqrt{\epsilon'} |E_0|^2 (\vec{n} \cdot \hat{e}_y)} \right| = \boxed{\left| \frac{E_0''}{E_0} \right|^2} = R$$



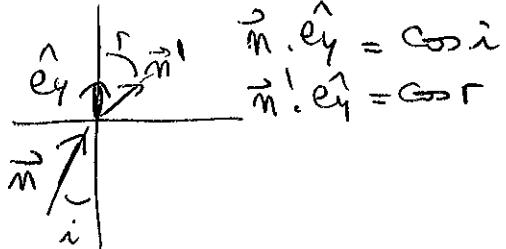
So for R the formula is the "obvious" one since both incident and reflected are in the same medium.

However, for T the formula will be more complex.

$$T = \left| \frac{\vec{S} \cdot \hat{e}_y}{\vec{S} \cdot \hat{e}_y} \right| = \left| \frac{\frac{1}{2} \sqrt{\epsilon'} |E_0'|^2 (\vec{n}' \cdot \hat{e}_y)}{\frac{1}{2} \sqrt{\epsilon'} |E_0|^2 (\vec{n} \cdot \hat{e}_y)} \right|$$

$$T = \left\{ \frac{\epsilon' \mu}{\epsilon \mu'} \right\} \left| \frac{E_0'}{E_0} \right|^2 \left| \frac{(\vec{n} \cdot \hat{e}_y)}{(\vec{n}' \cdot \hat{e}_y)} \right|^2$$

$$\left\{ \frac{\epsilon' \mu}{\epsilon \mu'} \right\} = \left\{ \frac{\epsilon' \mu'}{\epsilon \mu} \right\} \cdot \frac{\mu}{\mu'} \\ \frac{m'}{m}$$



$$\vec{n} \cdot \hat{e}_y = \cos i \\ \vec{n}' \cdot \hat{e}_y = \cos r$$

$$T = \frac{m'}{m} \frac{\mu}{\mu'} \left| \frac{\cos r}{\cos i} \right|^2 \left| \frac{E_0'}{E_0} \right|^2$$

) Plotting these results it can be checked that $R + T = 1$.

Consider the special case $i=0$. Then $r=0$ and $\frac{\cos r}{\cos i} = 1$.

Consider also $\mu=\mu'=1$, and $m=1$.

Then,

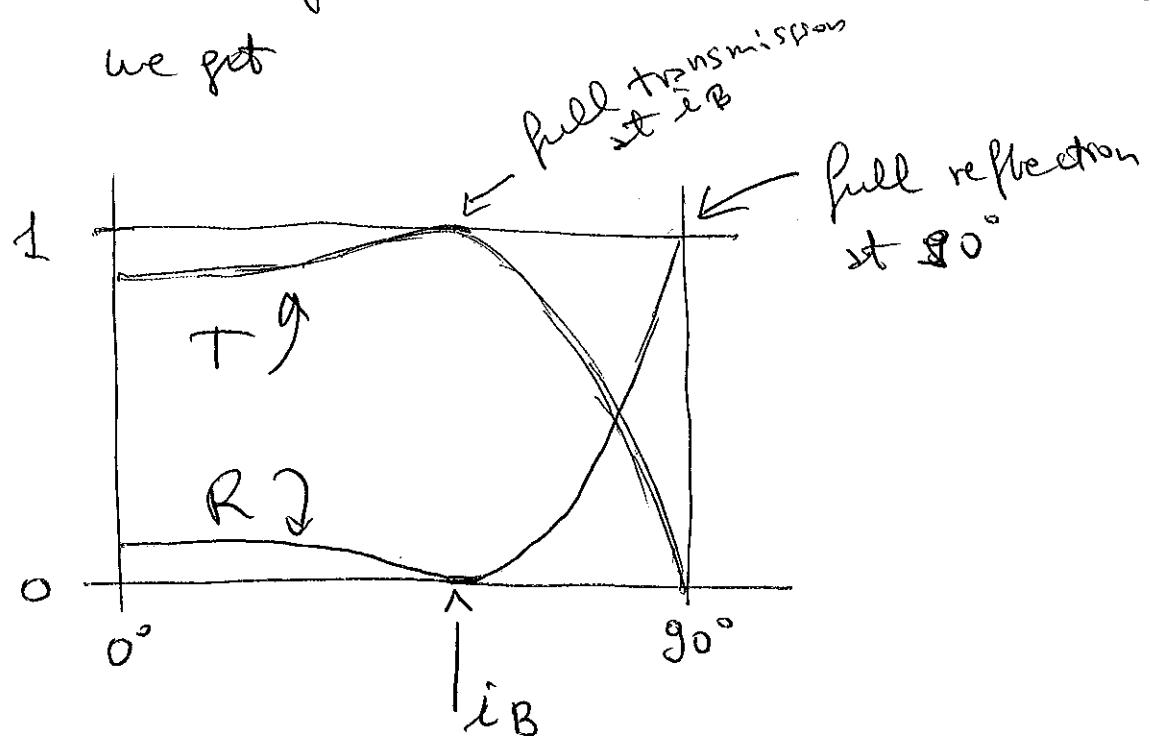
$$R \stackrel{i=0}{=} \left| \frac{(m')^2 - \sqrt{(m')^2}}{(m')^2 + \sqrt{(m')^2}} \right|^2 = \left| \frac{m' - 1}{m' + 1} \right|^2$$

$$T = \frac{m'}{m} \left| \frac{2m'}{(m')^2 + \sqrt{(m')^2}} \right|^2 = 4m' \left| \frac{1}{m' + 1} \right|^2$$

$$R+T = \frac{1}{(m'+1)^2} [(m'-1)^2 + 4m'] = \frac{m'^2 - 2m' + 1 + 4m'}{(m'+1)^2} = \frac{m'^2 + 2m' + 1}{(m'+1)^2} = 1$$

Plotting the results (see Fig. 9.17 of Griffiths, second edition)

) we get



)

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—○—