Show that Eq. (1.17) (page 35, top) is correct, deducing it from the Poisson Eq. (1.28), following the steps of page 35.

Let us try to see if $\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$ works by acting on it with ∇^2. Since the integrand has a singularity at $\vec{x}=\vec{x}'$, we have to be careful. For this reason we avoid the divergence of the integrand by adding an artificial constant "a" to avoid problemas. At the end of the calculation we have to take the limit $a\to 0$. So instead of the integral above we will use

$$\Phi_a(\vec{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')\,d^3x'}{\sqrt{(\vec{x}-\vec{x}')^2+a^2}}$$

no divergence of integrand since denominator is always $\neq 0$.

$$\nabla^2\Phi_a(\vec{x}) =$$

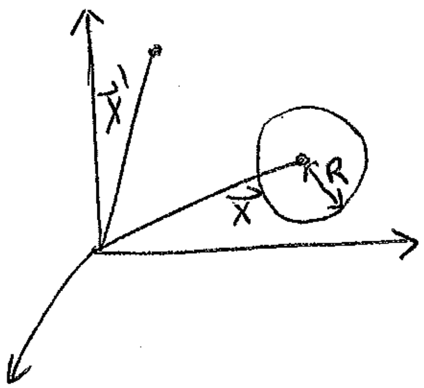
$$=\frac{1}{4\pi\epsilon_0}\int\rho(\vec{x}')\,\underbrace{\nabla^2\left(\frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+a^2}}\right)}\,d^3x'$$

It can be shown (see next pages) that this is

$$\frac{-3a^2}{[(\vec{x}-\vec{x}')^2+a^2]^{5/2}}$$

for instance by explicit calculation with cartesian coordinates.

$$=-\frac{1}{4\pi\epsilon_0}\int\rho(\vec{x}')\,\frac{3a^2}{[(\vec{x}-\vec{x}')^2+a^2]^{5/2}}\,d^3x' \qquad (1.30)$$



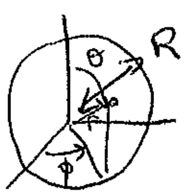
Consider R such that $a \ll R$.
On the outside of the sphere of radius R, $|\vec{x}-\vec{x}'|$ is always larger than a. Then,

$$\frac{3a^2}{[(\vec{x}-\vec{x}')^2+a^2]^{5/2}} \xrightarrow[a\to 0]{} 0$$

in the outside

(we are assuming $\rho(\vec{x}')$ is a "well-behaved" function)

Thus, only the integral over the "little" sphere is of relevance. We can then locate our system of coordinates to have the origin at $\vec{x}$. The integral we must solve is then:

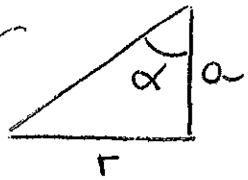


$$\left(\frac{-1}{4\pi\epsilon_0}\right)4\pi\int_0^R \frac{\rho(\vec{x}')\ 3a^2\ dr\, r^2}{(r^2+a^2)^{5/2}} \approx$$

assuming $\rho(\vec{x}')$ changes "slowly" at $\vec{x}$, which is the new origin of coordinates. This is why we need R to be "small" (but $\gg a$), i.e. to take ρ out of the integral.

$$\approx -\frac{1}{\epsilon_0}\rho(\vec{0})\underbrace{\int_0^R \frac{3a^2\, r^2 dr}{(r^2+a^2)^{5/2}}}_{}$$

We have to do this integral somehow (see back)



$$r = a \tan\alpha$$

$$\sqrt{r^2+a^2} = \frac{a}{\cos\alpha} \; ; \quad (r^2+a^2)^{5/2} = \frac{a^5}{\cos^5\alpha}$$

$$dr = a\, d(\tan\alpha)$$

$$= a\, d\left(\frac{\sin\alpha}{\cos\alpha}\right) = a\left[\frac{d(\sin\alpha)}{\cos\alpha} + \sin\alpha\, d\left(\frac{1}{\cos\alpha}\right)\right]$$

$$= a\left[\frac{\cancel{\cos\alpha}\, d\alpha}{\cancel{\cos\alpha}} + \sin\alpha \left(\frac{-1}{\cos^2\alpha}\right)(-\sin\alpha)\, d\alpha\right]$$

$$= a\, \frac{d\alpha}{\cos^2\alpha}$$

$$\int_0^R \frac{dr\, r^2}{(r^2+a^2)^{5/2}} = \int a \frac{d\alpha}{\cos^2\alpha}\, \frac{a^2\tan^2\alpha\, \cos^5\alpha}{a^5} = \int \frac{1}{a^2}\, d\alpha\, \sin^2\alpha \cos\alpha$$

$$= \frac{1}{a^2}\, \frac{\sin^3\alpha}{3}\Bigg|_0^{\alpha^R} = \frac{1}{a^2}\,\frac{\sin^3\alpha_R}{3}$$

← if $r=R$, $\alpha^R = \tan^{-1}(R/a)$

← Just quote this result.

if $r=0$, $\alpha=0$

If $\frac{R}{a}$ is large, $\alpha^R \approx \pi/2$

and $\sin^3\alpha_R \approx 1$

Then

$$\lim_{a\to 0}\int_0^R \frac{3a^2 r^2 dr}{(r^2+a^2)^{5/2}} = \lim_{a\to 0} 3a^2\, \frac{1}{a^2\, 3} = 1$$

Key result: this integral is indep. of R and a.

Switching back the origin of coordinates to $\vec{x}$.

Then $$\boxed{\lim_{a\to 0} \nabla^2 \Phi_a(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}}$$

The singular nature of $\nabla^2\left(\frac{1}{r}\right)$ can be formally

Extra calculation needed along the way

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{\sqrt{(\vec{x}-\vec{x}')^2+a^2}} =$$

$$= \frac{\partial}{\partial x}\left[\left(-\frac{1}{2}\right)\left[(\vec{x}-\vec{x}')^2+a^2\right]^{-3/2}\underbrace{\frac{\partial}{\partial x}(x-x')^2}_{2(x-x')}\right] + \cdots$$

$$= -\frac{\partial}{\partial x}\frac{(x-x')}{\left[(\vec{x}-\vec{x}')^2+a^2\right]^{3/2}} + \cdots = -\frac{1}{[\ldots]^{3/2}} + \frac{3}{2}\frac{(x-x')\;2(x-x')}{[\quad]^{5/2}} + \cdots$$

$$= -\frac{3}{[\ldots]^{3/2}} + 3\frac{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]}{[\ \ldots\ ]^{5/2}}$$

$$= \frac{1}{[\ldots]^{3/2}}\left[\frac{-3\left(|\vec{x}-\vec{x}'|^2+a^2\right)+3|\vec{x}-\vec{x}'|^2}{[\ldots]}\right] = \frac{-3a^2}{\left[(\vec{x}-\vec{x}')^2+a^2\right]^{5/2}}$$