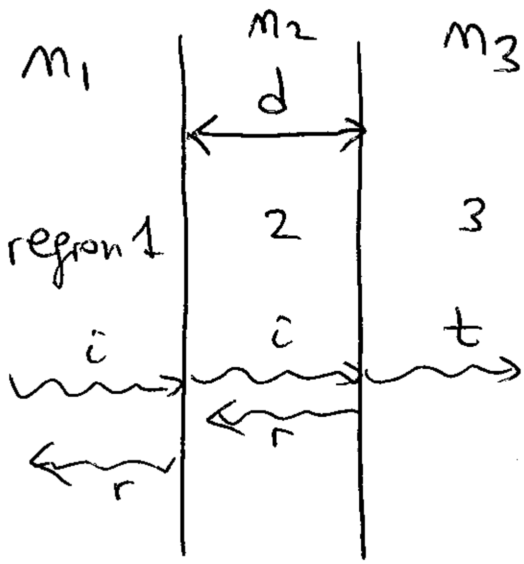
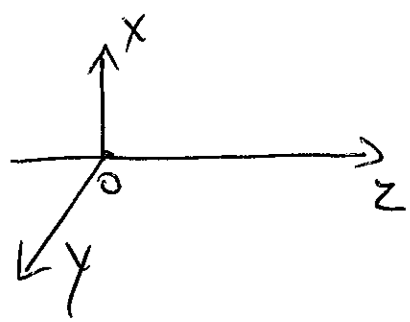
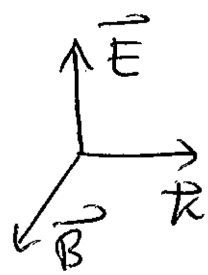
Problem 7.2 (b)





The wavevectors $\vec{k}$'s all point along the z axis. The $\vec{E}$ field are assumed to point along x and $\vec{B}$ fields along y.



The electric fields are:

$$E_1 = E_1^i e^{ik_1 z} + E_1^r e^{-ik_1 z}$$

$$E_2 = E_2^i e^{ik_2 z} + E_2^r e^{-ik_2 z}$$

$$E_3 = E_3^t e^{ik_3 z}$$

By continuity at the two surfaces we get:

$\underline{z=0}$
$$\boxed{E_1^i + E_1^r = E_2^i + E_2^r}$$

$\underline{z=d}$
$$\boxed{E_2^i e^{ik_2 d} + E_2^r e^{-ik_2 d} = E_3^t e^{ik_3 d}}$$

The other equations to be used arise from the continuity of the magnetic fields at the interfaces.

From $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ for a plane wave with time dependence $e^{-i\omega t}$ we get:

$$\vec{B} = \frac{i}{\omega}(\nabla \times \vec{E}) = \frac{i}{\omega} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \nabla_x & \nabla_y & \nabla_z \\ E & 0 & 0 \end{vmatrix} \underset{\uparrow}{=} \frac{i}{\omega} \nabla_z E \, \hat{j}$$

$\nabla_y E = 0$

$$B_1 = \frac{i}{\omega}\left(E_1^i \, ik_1 \, e^{ik_1 z} - E_1^r \, ik_1 \, e^{-ik_1 z}\right)$$
$$= \frac{i^2 k_1}{\omega}\left(E_1^i e^{ik_1 z} - E_1^r e^{-ik_1 z}\right)$$

$$B_2 = \frac{i^2 k_2}{\omega}\left(E_2^i e^{ik_2 z} - E_2^r e^{-ik_2 z}\right)$$

$$B_3 = \frac{i^2 k_3}{\omega} E_3^t e^{ik_3 z}$$

At $\underline{z=0}$:

Note that the continuity is that of $\vec{H}$, thus we divide by the respective μ's.

$$\frac{k_1}{\mu_1}\left(E_1^i - E_1^r\right) = \frac{k_2}{\mu_2}\left(E_2^i - E_2^r\right)$$

$$\boxed{E_1^i - E_1^r = \frac{k_2\mu_1}{k_1\mu_2}\left(E_2^i - E_2^r\right)}$$

$$\xrightarrow{(7.4)} \frac{\sqrt{\mu_2\epsilon_2}\,\cancel{\omega}}{\sqrt{\mu_1\epsilon_1}\,\cancel{\omega}} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_0\epsilon_0}}\sqrt{\frac{\mu_0\epsilon_0}{\mu_1\epsilon_1}} \overset{(7.5)}{=} \frac{n_2}{n_1}$$

At $\underline{z=d}$:

$$\frac{k_2}{\mu_2}\left(E_2^i e^{ik_2d} - E_2^r e^{-ik_2d}\right) = \frac{k_3}{\mu_3} E_3^t e^{ik_3d}$$

$$\boxed{\left(E_2^i e^{ik_2d} - E_2^r e^{-ik_2d}\right) = \frac{n_3\mu_2}{n_2\mu_3} E_3^t e^{ik_3d}}$$

Adding the second and fourth: define $\tilde{n}_i = \frac{n_i}{\mu_i}$

$$2E_2^i e^{ik_2 d} = E_3^t e^{ik_3 d}\left(1 + \frac{\tilde{n}_3}{\tilde{n}_2}\right)$$

$$E_2^i = \frac{1}{2}\left(1 + \frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3-k_2)d} E_3^t$$

Substracting the second and fourth:

$$2E_2^r e^{-ik_2 d} = E_3^t e^{ik_3 d}\left(1 - \frac{\tilde{n}_3}{\tilde{n}_2}\right)$$

$$E_2^r = \frac{1}{2}\left(1 - \frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3+k_2)d} E_3^t$$

Adding first and third:

$$2E_1^i = E_2^i\left(1 + \frac{\tilde{n}_2}{\tilde{n}_1}\right) + E_2^r\left(1 - \frac{\tilde{n}_2}{\tilde{n}_1}\right)$$

$$= \frac{1}{2}\left(1 + \frac{\tilde{n}_3}{n_2}\right) e^{i(k_3-k_2)d} E_3^t \left(1 + \frac{\tilde{n}_2}{\tilde{n}_1}\right)$$

$$+ \frac{1}{2}\left(1 - \frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3+k_2)d} E_3^t \left(1 - \frac{\tilde{n}_2}{n_1}\right)$$

$$\cancel{E_1^i = \frac{1}{4} e^{i(k_3-k_2)d} E_3^t \left[\left(1 + \frac{n_3}{n_2}\right)\left(1 + \frac{n_2}{n_1}\right) + \left(1 - \frac{n_3}{n_2}\right)\left(1 - \frac{n_2}{n_1}\right)\right]}$$

$$\boxed{\frac{E_3^t}{E_1^i} = \frac{4}{\left(1+\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3-k_2)d}\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3+k_2)d}\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right)}}.$$

If $\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = 1$, then $k_3 = k_2$,

and $\frac{E_3^t}{E_1^i} = 1$ as expected.

Substracting first and third:

$$2E_1^r = E_2^i\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right) + E_2^r\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)$$

$$= \frac{1}{2}\left(1+\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3-k_2)d} E_3^t\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right)$$

$$+\frac{1}{2}\left(1-\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3+k_2)d} E_3^t\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)$$

$$\boxed{\frac{E_3^t}{E_1^r} = \frac{4}{\left(1+\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3-k_2)d}\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{\tilde{n}_3}{\tilde{n}_2}\right)e^{i(k_3+k_2)d}\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)}}$$

If $\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = 1$, $k_3 = k_2$, it diverges as it should.

$$\boxed{\frac{E_1^r}{E_1^i} = \frac{\left(1+\frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3-k_2)d}\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3+k_2)d}\left(1+\frac{n_2}{n_1}\right)}{\left(1+\frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3-k_2)d}\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{\tilde{n}_3}{\tilde{n}_2}\right) e^{i(k_3+k_2)d}\left(1-\frac{n_2}{n_1}\right)}}$$

If $\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = 1$, it is 0 as expected.

Suppose $n_3 = 1$ ($\tilde{n}_3 = \frac{1}{\mu_0}$) and we want E_1^r to cancel:

$$E_1^r = e^{ik_3d}\left[\left(1+\frac{1}{\mu_0\tilde{n}_2}\right)\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right)e^{-ik_2d} + \left(1-\frac{1}{\mu_0\tilde{n}_2}\right)\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)e^{ik_2d}\right]$$

$$= 0$$

$$\overbrace{\left[\left(1+\frac{1}{\mu_0\tilde{n}_2}\right)\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{1}{\mu_0\tilde{n}_2}\right)\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)\right]}^{(a)}\cos(k_2 d)$$

$$\underbrace{\left[-\left(1+\frac{1}{\mu_0\tilde{n}_2}\right)\left(1-\frac{\tilde{n}_2}{\tilde{n}_1}\right) + \left(1-\frac{1}{\mu_0\tilde{n}_2}\right)\left(1+\frac{\tilde{n}_2}{\tilde{n}_1}\right)\right]}_{(b)} i\sin(k_2 d) = 0$$

ⓐ $= \left(1 + \cancel{\frac{1}{\mu_0\tilde{n}_2}} - \cancel{\frac{\tilde{n}_2}{\tilde{n}_1}} - \frac{1}{\mu_0\tilde{n}_1}\right) + \left(1 - \cancel{\frac{1}{\mu_0\tilde{n}_2}} + \cancel{\frac{\tilde{n}_2}{\tilde{n}_1}} - \frac{1}{\mu_0\tilde{n}_1}\right)$

$= 2\left(1 - \frac{1}{\mu_0\tilde{n}_1}\right)$

ⓑ $= -\left(\cancel{1} + \frac{1}{\mu_0\tilde{n}_2} - \frac{\tilde{n}_2}{\tilde{n}_1} - \frac{1}{\mu_0\tilde{n}_1}\right) + \left(\cancel{1} - \frac{1}{\mu_0\tilde{n}_2} + \frac{\tilde{n}_2}{\tilde{n}_1} - \frac{1}{\mu_0\tilde{n}_1}\right)$

$= \frac{-2}{\mu_0\tilde{n}_2} + \frac{2\tilde{n}_2}{\tilde{n}_1} = 2\left(\frac{\mu_0\tilde{n}_2^2 - \tilde{n}_1}{\mu_0\tilde{n}_2\tilde{n}_1}\right)$

For $2\left(1 - \frac{1}{\mu_0\tilde{n}_1}\right)\cos(k_2 d) + 2\left(\frac{\mu_0\tilde{n}_2^2 - \tilde{n}_1}{\mu_0\tilde{n}_2\tilde{n}_1}\right) i\sin(k_2 d) = 0$

to be true note that both real and imaginary must cancel. Since $n_1 > 1$, then

$\boxed{\cos(k_2 d) = 0}$ is needed

Since $\cos(k_2 d) = 0$ means $\sin(k_2 d) = 1$, then we must also request:

$\boxed{\mu_0\tilde{n}_2^2 - \tilde{n}_1 = 0} \rightarrow \mu_0 \frac{n_2 n_2}{\mu_2^2} - \frac{n_1}{\mu_1} = 0$

Thus, we need : $\boxed{\begin{aligned} n_2 &= \sqrt{n_1 \frac{\mu_2^2}{\mu_0\mu_1}} \\ d_0 k_2 &= (n + 1/2)\pi \end{aligned}}$ $(n = 0, 1, 2, \ldots)$

with $k_2 \underset{\uparrow}{=} \frac{n_2 \omega}{c}$ or $\boxed{d = \frac{(n + 1/2)\pi c}{n_2 \omega_0}} \Longleftarrow$

(7.5)