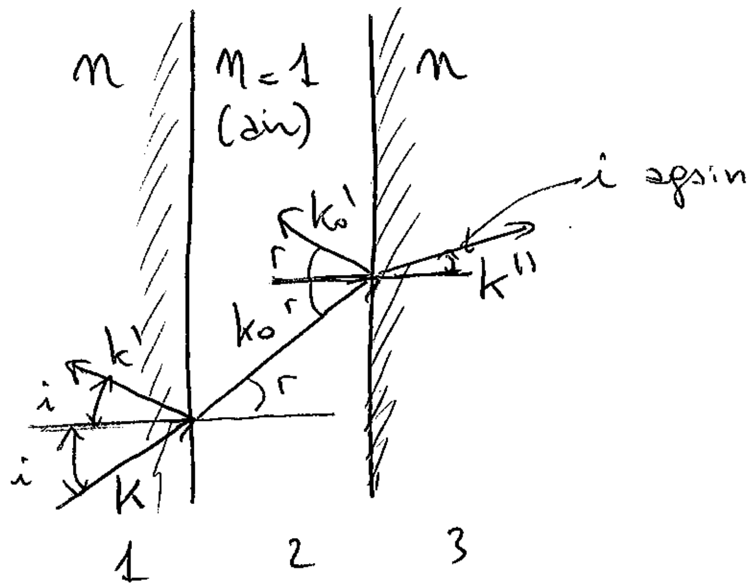
Problem 7.3 (a)



$$\vec{E}_1 = \vec{E}_1^{\,i} e^{i\vec{k}\cdot\vec{x}} + \vec{E}_1^{\,r} e^{-i\vec{k}'\cdot\vec{x}} \qquad (|\vec{k}'| = |\vec{k}|)$$

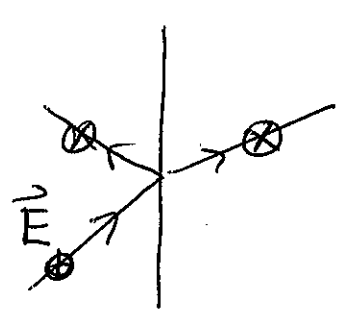
$$\vec{E}_2 = \vec{E}_0^{\,i} e^{i\vec{k}_0\cdot\vec{x}} + \vec{E}_0^{\,r} e^{-i\vec{k}_0'\cdot\vec{x}} \qquad (|\vec{k}_0'| = |\vec{k}_0|)$$

$$\vec{E}_3 = \vec{E}_3^{\,t} e^{i\vec{k}''\cdot\vec{x}} \qquad (\vec{k}'' = \vec{k})$$

$$n \sin i = \sin r$$

$$\text{or } \cos r = \sqrt{1 - n^2 \sin^2 i}$$

Let us consider an incoming wave with $\vec{E} \perp$ to the plane of incidence, similarly as in Problem 7.2.



We should use as boundary condition that tangencial component of $\vec{E}$ be continuous. But $\vec{E}$ is already tangencial since it points along the x axis. Thus, it is sufficient to ask:

$$\boxed{E_1^i + E_1^r = E_0^i + E_0^r} \quad \text{at } z=0$$

At $z=d$, note that the $e^{i\vec{k}\cdot\vec{x}}$ phase factor may matter. We know that $\vec{k}_{\parallel}$ are continuous at the interfaces. But the z components must be kept:

$$E_0^i \, e^{i\underbrace{k_0^z d}_{k_0 d\cos r \equiv \phi}} + E_0^r \, e^{-ik_0^z d} = E_3^t \, e^{i\underbrace{k^z d}_{kd\cos i \equiv \varphi}}$$

$$\boxed{E_0^i e^{i\phi} + E_0^r e^{-i\phi} = E_3^t e^{i\varphi}}$$

Note: As we did in Problem 7.2, let us consider now the continuity of $\vec{B}$ (normal to surface) at interface:
and the continuity of $\vec{H}$ (// to surface)

$$\vec{B} = \frac{i}{\omega}(\nabla \times \vec{E}) = \frac{i}{\omega}\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \nabla_x & \nabla_y & \nabla_z \\ E & 0 & 0 \end{vmatrix} = \frac{i}{\omega}\left(\hat{e}_y \nabla_z E - \hat{e}_z \nabla_y E\right)$$

$$\vec{E} = E\,\hat{e}_x \underset{\substack{\uparrow \\ \text{first region.}}}{=} \left(E_1^i\, e^{i\vec{k}.\vec{x}} + E_1^r\, e^{+i\vec{k}'.\vec{x}}\right)\hat{e}_x$$

$$\nabla_z E = \frac{d}{dz}E = E_1^i\, ik_z\, e^{i\vec{k}.\vec{x}} + E_1^r\, i\underbrace{k_z'}_{-k_z} e^{i\vec{k}'.\vec{x}}$$

$$= ik_z\left(E_1^i e^{i\vec{k}.\vec{x}} - E_1^r e^{i\vec{k}'.\vec{x}}\right)$$

~~For second region~~:

$$\nabla_y E = \frac{d}{dy}E = E_1^i\, ik_y e^{i\vec{k}.\vec{x}} + E_1^r\, i\underbrace{k_y'}_{k_y} e^{i\vec{k}'.\vec{x}}$$

$$= ik_y\left(E_1^i e^{i\vec{k}.\vec{x}} + E_1^r e^{i\vec{k}'.\vec{x}}\right)$$

1. It is the "normal" component of $\vec{B}$ that must be continuous i.e. the component along the z axis:

$$\left(\frac{-i}{\omega}\right) ik_y\left(E_1^i e^{i\vec{k}.\vec{x}} + E_1^r e^{i\vec{k}'.\vec{x}}\right) = \left(\frac{-i}{\omega}\right) ik_y^0\left(E_0^i e^{i\vec{k}_0.\vec{x}} + E_0^r e^{i\vec{k}_0'.\vec{x}}\right)$$

Consider $z=0$:

$$e^{i\vec{k}\cdot\vec{x}} = e^{i(k_z z + k_y y)} \overset{z=0}{\underline{\underline{\checkmark}}} e^{i k_y y}$$

$$e^{i\vec{k}'\cdot\vec{x}} = e^{i k_y y}$$

$$e^{i\vec{k}_0\cdot\vec{x}} = e^{i k_y^0 y}$$ ~~But $k_y^0 = k_y$ since $\sin i = \sin r$~~

$$e^{i\vec{k}_0'\cdot\vec{x}} = e^{i k_y^0 y}$$

Then:

$$k_y\left(E_1^i e^{i k_y y} + E_1^r e^{i k_y y}\right) = k_y^0\left(E_0^i e^{i k_y^0 y} + E_0^r e^{i k_y^0 y}\right)$$

$$k_y = k \sin i$$

$$k_y^0 = k^0 \sin r = k \sin i$$

But $k^0 = k \dfrac{\sin i}{\sin r}$ since tangencial components of $\vec{k}$'s must be the same.

Then:

$$E_1^i + E_1^r = E_0^i + E_0^r$$

which we already found before.

Then, considering the normal continuity of $\vec{B}$ is $\underline{\underline{\text{redundant}}}$.

② Consider now the continuity of the tangencial component of $\vec{H}$: then we must focus on the $\hat{e}_y$ component of $\vec{B}$.

$$\frac{i}{\omega}\,\frac{ik_z}{\mu}\left(E_1^i e^{i\vec{k}\cdot\vec{x}} - E_1^r e^{i\vec{k}'\cdot\vec{x}}\right) = \frac{i}{\omega}\,\frac{ik_z^o}{\underset{\uparrow \text{air}}{\mu'}}\left(E_o^i e^{i\vec{k}_o\cdot\vec{x}} - E_o^r e^{i\vec{k}_o'\cdot\vec{x}}\right)$$

There is a phase factor with the tangencial $\vec{k}$'s that is common to all. And at $z=0$, the z component of $\vec{k}$ vanishes as well. Then:

$$\frac{k_z}{\mu}\left(E_1^i - E_1^r\right) = \frac{k_z^o}{\mu'}\left(E_o^i - E_o^r\right)$$

$$k_z = k\cos i \qquad \text{and} \qquad \frac{k^o}{k} = \frac{n^o}{n} = \frac{1}{n}$$

$$k_z^o = k^o \cos r$$

Thus:

$$\boxed{\frac{\mu'}{\mu}\, n\left(E_1^i - E_1^r\right) = \frac{\cos r}{\cos i}\left(E_o^i - E_o^r\right)}$$

Now consider $z=d$:

$$\frac{k_z^o}{\underset{\nwarrow \text{air}}{\mu'}}\left(E_o^i \underbrace{e^{ik_z^o d}}_{\text{we called this } \phi \text{ before}} - E_o^r e^{-ik_z^o d}\right) = \frac{k_z}{\mu} E_3^t e^{ik_z d}$$

$$\frac{\mu}{\mu'} \underbrace{\left(\frac{k_z^0}{k_z}\right)}_{\frac{\cos r}{\cos i} \frac{1}{n}} \left(E_0^i e^{i\phi} - E_0^r e^{-i\phi}\right) = E_3^t e^{ik_z d}$$

$$\boxed{\frac{\mu}{\mu'} \frac{1}{n} \frac{\cos r}{\cos i} \left(E_0^i e^{i\phi} - E_0^r e^{-i\phi}\right) = E_3^t e^{i\overbrace{k_z d}^{\varphi}}}$$

So, the four equations are:

$$\boxed{\begin{aligned}
&E_1^i + E_1^r = E_0^i + E_0^r \\
&E_0^i e^{i\phi} + E_0^r e^{-i\phi} = E_3^t e^{i\varphi} \\
&\frac{\mu'}{\mu} n \left(E_1^i - E_1^r\right) = \frac{\cos r}{\cos i}\left(E_0^i - E_0^r\right) \\
&\frac{\mu}{\mu'}\frac{1}{n}\frac{\cos r}{\cos i}\left(E_0^i e^{i\phi} - E_0^r e^{-i\phi}\right) = E_3^t e^{i\varphi}
\end{aligned}}$$

Let us define $\alpha = \frac{\cos r}{\cos i}\frac{\mu}{\mu'}\frac{1}{n}$. Then, we have:

$$E_1^i + E_1^r = E_0^i + E_0^r \qquad (1)$$

$$E_0^i e^{i\phi} + E_0^r e^{-i\phi} = E_3^t e^{i\varphi} = \tilde{E}_3^t \qquad (2)$$

$$E_1^i - E_1^r = \alpha\left(E_0^i - E_0^r\right) \qquad (3)$$

$$\alpha\left(E_0^i e^{i\phi} - E_0^r e^{-i\phi}\right) = E_3^t e^{i\varphi} = \tilde{E}_3^t \qquad (4)$$

(1) + (3) gives: $E_1^i = \frac{1}{2}\left(E_0^i(1+\alpha) + E_0^r(1-\alpha)\right) = \frac{E_0^i}{2}\left((1+\alpha) + \frac{E_0^r}{E_0^i}(1-\alpha)\right)$

(1) − (3) gives: $E_1^r = \frac{1}{2}\left(E_0^i(1-\alpha) + E_0^r(1+\alpha)\right) = \frac{E_0^i}{2}\left((1-\alpha) + \frac{E_0^r}{E_0^i}(1+\alpha)\right)$

From ② and ④:

$$E_0^i e^{i\phi} + E_0^r e^{-i\phi} = \alpha\left(E_0^i e^{i\phi} - E_0^r e^{-i\phi}\right)$$

$$\hookrightarrow \frac{E_0^r}{E_0^i} = \frac{(\alpha-1)e^{i\phi}}{(\alpha+1)e^{-i\phi}}$$

Thus:

$$\frac{E_1^r}{E_1^i} = \frac{1-\alpha + \frac{E_0^r}{E_0^i}(1+\alpha)}{1+\alpha + \frac{E_0^r}{E_0^i}(1-\alpha)} = \frac{1-\alpha + \frac{(\alpha-1)}{(1+\alpha)}e^{+2i\phi}(1+\alpha)}{1+\alpha + \frac{(\alpha-1)}{(1+\alpha)}e^{+2i\phi}(1-\alpha)}$$

$$= \frac{(1-\alpha^2)(1 - e^{+2i\phi})}{(1+\alpha)^2 - (1-\alpha)^2 e^{+2i\phi}}$$

$$\boxed{R = \left|\frac{E_1^r}{E_1^i}\right|^2 = \left|\frac{(1-\alpha^2)(1-e^{+2i\phi})}{(1+\alpha)^2 - (1-\alpha)^2 e^{+2i\phi}}\right|^2}$$

$$\frac{\tilde{E}_3^t}{E_1^i} = \frac{\tilde{E}_3^t/E_0^i}{E_1^i/E_0^i} = \frac{e^{i\phi} + \frac{E_0^r}{E_0^i}e^{-i\phi}}{\frac{1}{2}\left[(1+\alpha) + \frac{E_0^r}{E_0^i}(1-\alpha)\right]} = \frac{e^{i\phi} - \frac{(1-\alpha)}{(1+\alpha)}e^{2i\phi}e^{-i\phi}}{\frac{1}{2}\left[(1+\alpha) + (-)\frac{(1-\alpha)}{(1+\alpha)}e^{2i\phi}(1-\alpha)\right]}$$

$$= \frac{e^{i\phi}\,4\alpha}{(1+\alpha)^2 - (1-\alpha)^2 e^{2i\phi}} \;; \quad \boxed{T = \left|\frac{\tilde{E}_3^t}{E_1^i}\right|^2 = \frac{16\alpha^2}{\left|(1+\alpha)^2 - (1-\alpha)^2 e^{2i\phi}\right|^2}}$$

Is $R + T = 1$?

$$\left|(1+\alpha)^2 - (1-\alpha)^2 e^{2i\phi}\right|^2 = \left[(1+\alpha)^2 - (1-\alpha)^2 e^{2i\phi}\right]\left[(1+\alpha)^2 - (1-\alpha)^2 e^{-2i\phi}\right] =$$

$$= (1+\alpha)^4 + (1-\alpha)^4 - (1-\alpha)^2(1+\alpha)^2 \underbrace{\left(e^{2i\phi} + e^{-2i\phi}\right)}_{2\cos(2\phi)}$$

$$= (1+\alpha)^4 + (1-\alpha)^4 - (1-\alpha)^2(1+\alpha)^2\, 2\cos(2\phi) \leftarrow \text{denominator}$$

The numerator is:

$$16\alpha^2 + (1-\alpha^2)^2 \underbrace{\left(1 - e^{2i\phi}\right)\left(1 - e^{-2i\phi}\right)}_{2 - \underbrace{\left(e^{2i\phi} + e^{-2i\phi}\right)}_{2\cos(2\phi)}} = 16\alpha^2 + (1-\alpha^2)^2\, 2\left(1 - \cos(2\phi)\right)$$

$$= 16\alpha^2 + 2(1-\alpha^2)^2 - \underbrace{(1-\alpha^2)^2}_{[(1-\alpha)(1+\alpha)]^2 = (1-\alpha)^2(1+\alpha)^2} 2\cos 2\phi$$

i.e. the same coeff. of $2\cos(2\phi)$ in denominator.

Is $(1+\alpha)^4 + (1-\alpha)^4 \overset{?}{=} 16\alpha^2 + 2(1-\alpha^2)^2$

$$(1+\alpha)^4 = \left((1+\alpha)^2\right)^2 = (1 + 2\alpha + \alpha^2)^2 = 1 + 2(2\alpha + \alpha^2) + (2\alpha + \alpha^2)^2$$

$$= 1 + 4\alpha + 2\alpha^2 + 4\alpha^2 + 4\alpha^3 + \alpha^4$$

$$(1-\alpha)^4 = 1 - 4\alpha + 2\alpha^2 + 4\alpha^2 - 4\alpha^3 + \alpha^4$$

$$(1+\alpha)^4 + (1-\alpha)^4 = 2 + 12\alpha^2 + 2\alpha^4$$

$$16\alpha^2 + 2(1-\alpha^2)^2 = 16\alpha^2 + 2(1-2\alpha^2+\alpha^4)$$
$$= 16\alpha^2 + 2 - 4\alpha^2 + 2\alpha^4$$
$$= 2 + 12\alpha^2 + 2\alpha^4 \quad \underline{\underline{\text{equal!}}}$$

Then $\boxed{R + T = 1}$.

Note that since the ratios of "powers" we are asked for are in the same medium

 and 

then ratios of electric fields^2 are ratios of power (i.e. for 2 different media, the ratios of power would ~~also~~ need ratios of μ's and/or ϵ's)