Jackson 9.3

From (2.27) we see that at large distances (i.e. those relevant for radiation)

$$\phi(r,\theta,\phi) \approx \frac{3VR^2}{2r^2}\cos\theta$$

From (4.10), $\phi_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$.

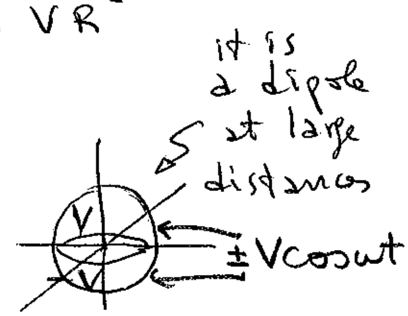
By symmetry $\vec{p} = p\hat{e}_z$, $\vec{p}\cdot\vec{x} = pr\cos\theta$. Then:

$$\phi_{dipole} = \frac{1}{4\pi\epsilon_0}\,p\frac{r\cos\theta}{r^3} = \frac{3VR^2}{2r^2}\cos\theta$$

Thus: $p = \frac{3VR^2}{2}4\pi\epsilon_0 = 6\pi\epsilon_0 VR^2$

and including the time dependence:

$$p(\omega) = p\,e^{-i\omega t}$$



From (9.19):

Once we have $\vec{p}$, no need to get $\vec{A}$, just use generic formulas:

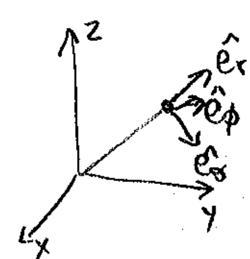
$$\vec{H} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\underbrace{(\vec{n}\times\vec{p})} = \boxed{\frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(-p\sin\theta)\,\hat{e}_\phi}$$

$$p\begin{vmatrix}\hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0\end{vmatrix} = -p\sin\theta\,\hat{e}_\phi$$

(we used $\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$)

From (9.19):

$$\vec{E} = Z_0(\vec{H}\times\vec{n}) = \boxed{Z_0\frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(-p\sin\theta)\underbrace{(\hat{e}_\phi\times\hat{e}_r)}_{\hat{e}_\theta}}$$



$$\underbrace{\vec{E}\times\vec{H}^*}_{\substack{\text{important for}\\ \text{time average}}} = Z_0\left(\frac{ck^2}{4\pi}\frac{1}{r}p\sin\theta\right)^2 \underbrace{\hat{e}_\theta\times\hat{e}_\phi}_{\hat{e}_r}$$

$$\frac{r^2}{2}\underset{\hat{e}_r}{\underset{\uparrow}{\vec{n}}}\cdot(\vec{E}\times\vec{H}^*) = \frac{1}{2}Z_0\left(\frac{ck^2}{4\pi}p\sin\theta\right)^2 = \frac{dP}{d\Omega}$$

$$Z_0=\sqrt{\frac{\mu_0}{\epsilon_0}}\,,\quad k=\frac{\omega}{c}\,,\quad \frac{1}{c}=\sqrt{\mu_0\epsilon_0}\quad (9.21)$$

$$\frac{dP}{d\Omega} = \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{c}{4\cancel{\pi}}\frac{\omega^2}{c^2}\,6\cancel{\pi}\,\epsilon_0 V R^2\sin\theta\right)^2$$

$$= \frac{1}{2}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{3}{2}\right)^2\frac{\omega^4}{c^2}\,\epsilon_0^2 V^2 R^4\sin^2\theta$$

$$= \boxed{\frac{9}{8}\frac{\epsilon_0}{c^3}V^2R^4\omega^4\sin^2\theta = \frac{dP}{d\Omega}}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{\epsilon_0^2}{c^2} = \frac{1}{c^3}\sqrt{\frac{\cancel{\mu_0}}{\epsilon_0}}\,\epsilon_0^2\frac{1}{\cancel{\sqrt{\mu_0\epsilon_0}}} = \frac{1}{c^3}\epsilon_0$$

$$P=\int d\Omega\left(\frac{dP}{d\Omega}\right) = \frac{9}{8}\frac{\epsilon_0}{c^3}V^2R^4\omega^4\underbrace{\int_0^{2\pi}d\phi}_{2\pi}\underbrace{\int_0^{\pi}\sin\theta\, d\theta\,\sin^2\theta}_{\int_0^\pi d\theta\,(\sin\theta-\sin\theta\cos^2\theta)}$$

$$\int_0^\pi d\theta\,(\sin\theta-\sin\theta\cos^2\theta) = (-\cos\theta)\Big|_0^\pi - \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^\pi$$

$$= 2+\frac{1}{3}(-2) = \frac{4}{3}$$

$$= \frac{9}{8}\frac{\epsilon_0}{c^3}V^2R^4\omega^4\,2\pi\,\frac{4}{3}$$

$$= \boxed{3\pi\frac{\epsilon_0}{c^3}V^2R^4\omega^4}$$

— o —