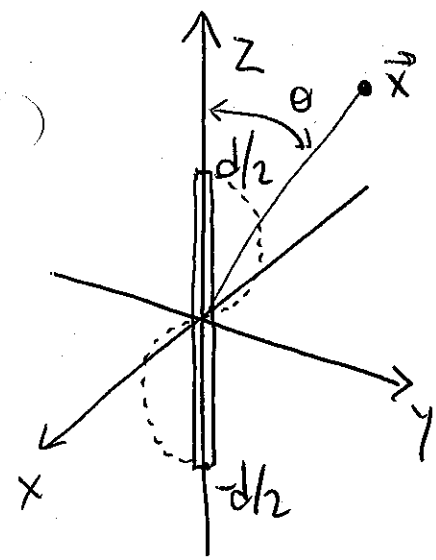
Jackson 9.16 (a)



We will write the current as:

$$\vec{J}(z) = \hat{e}_z \mathbf{I} \underbrace{\delta(x)\,\delta(y)}_{\text{"thin linear antenna"}} \sin(kz)\, \underbrace{\theta\left(\frac{d}{2} - |z|\right)}_{\text{to show explicitly that the antenna is in } [-d/2,\, d/2].}$$

Note that contrary to other cases solved before, at z=0 here J is always zero. Then likely dipolar contribution cancels

with $k = \frac{2\pi}{d}$ so that

$$\sin\left(k\frac{d}{2}\right) = \sin\left(-k\frac{d}{2}\right) = 0$$

Let us use the vector potential (9.8):

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}(\vec{x}')\, e^{-ik\,\vec{n}\cdot\vec{x}'}\, d^3x'$$

This is valid in the "radiation zone" which is what we are interested in.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \hat{e}_z \mathbf{I} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} \sin(kz')\, e^{-ikz'\cos\theta}\, dz'$$

in $\vec{n}\cdot\vec{x}'$, note that the δ functions reduce $\vec{x}'$ to just point along the z axis, i.e. $\vec{x}'$ becomes $z'\hat{e}_z$. And $\vec{n}$ is the unit vector in the $\vec{x}$ direction.

If we look at the real part of $e^{-ikz'\cos\theta}$ we get

$$\int_{-d/2}^{d/2} \sin(kz')\cos(kz'\cos\theta)\,dz'$$

but $\sin(kz')$ is odd under $z' \to -z'$
and $\cos(kz'\cos\theta)$ is even under $z' \to -z'$
thus, this integral cancells.

The imaginary part gives:

$$\int_{-d/2}^{d/2} \underbrace{\sin(kz')}_{\substack{\text{odd}\\ z'\to -z'}}\ \underbrace{\sin(kz'\cos\theta)}_{\substack{\text{odd}\\ z'\to -z'}}\,dz' = 2\int_{0}^{d/2} \sin(kz')\sin(kz'\cos\theta)\,dz'$$

Then:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\hat{e}_z I \frac{e^{ikr}}{r}\,2(-i)\int_0^{d/2} \sin(kz')\sin(kz'\cos\theta)\,dz'$$

Here we can use the trigonometric identity:

$$\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\sin(kz')\sin(kz'\cos\theta) = \frac{1}{2}\left[\cos[kz'(1-\cos\theta)] - \cos[kz'(1+\cos\theta)]\right]$$

$$\int_0^{d/2} \cos[kz'(1-\cos\theta)]\,dz' = \left.\frac{\sin[\overbrace{k}^{2\pi/d} z'(1-\cos\theta)]}{k(1-\cos\theta)}\right|_0^{d/2} =$$

$$= \frac{\sin[\pi(1-\cos\theta)]}{\underbrace{\frac{2\pi}{d}}_{k}(1-\cos\theta)} \underset{\sin(\pi-\alpha)=\sin\alpha}{=} \frac{\sin(\pi\cos\theta)}{k(1-\cos\theta)} .$$

$$\int_0^{d/2} \cos[kz'(1+\cos\theta)]\,dz' = \left.\frac{\sin[\overbrace{k}^{2\pi/d} z'(1+\cos\theta)]}{k(1+\cos\theta)}\right|_0^{d/2} = \frac{\sin[\pi(1+\cos\theta)]}{k(1+\cos\theta)} \underset{\substack{\sin(\pi+\alpha)\\=-\sin\alpha}}{=} -\frac{\sin(\pi\cos\theta)}{k(1+\cos\theta)} .$$

All together:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\hat{e}_z I \frac{e^{ikr}}{r} 2(-i)\frac{1}{2} \underbrace{\left[\frac{\sin(\pi(\cos\theta))}{k(1-\cos\theta)} + \frac{\sin(\pi(\cos\theta))}{k(1+\cos\theta)}\right]}_{\frac{\sin(\pi\cos\theta)}{k}\cdot\underbrace{\frac{(1+\cos\theta+1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}}_{\frac{2}{1-\cos^2\theta}=\frac{2}{\sin^2\theta}}}$$

$$= \boxed{\frac{\mu_0}{4\pi}\hat{e}_z I \frac{e^{ikr}}{kr}(-i)\, 2\, \frac{\sin(\pi\cos\theta)}{\sin^2\theta}.}$$

In the radiation zone we know that

$$\vec{H} = \frac{ik}{\mu_0}(\vec{n}\times\vec{A}) = \frac{i\cancel{k}}{\cancel{\mu_0}}\hat{e}_\phi \frac{\cancel{\mu_0}}{\underbrace{4\pi}_{2}} I \frac{e^{ikr}}{\cancel{k}r}(-i)\cancel{2}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}(-\cancel{\sin\theta})$$

$$= -\frac{\hat{e}_\phi I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}$$

$\vec{n} = \vec{e}_r$ and $\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$ (back cover Griffiths)

Then: $\vec{n}\times\vec{e}_z = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_\phi \\ 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = -\sin\theta\,\hat{e}_\phi$

$$\boxed{\vec{H} = -\hat{e}_\phi \frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}.}$$

The electric field is given by (8.19):

$$\vec{E} = Z_0(\vec{H}\times\vec{n}) = \frac{Z_0 I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}\underbrace{(-\hat{e}_\phi\times\hat{e}_r)}_{-e_\theta}$$

$$\vec{n}\cdot\vec{E}\times\vec{H}^* = Z_0\left(\frac{I}{2\pi}\right)^2\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\frac{1}{r^2}\underbrace{\vec{n}\cdot\underbrace{(\hat{e}_\theta\times\hat{e}_\phi)}_{\hat{e}_r}}_{1}$$

$$\frac{1}{2}r^2\vec{n}\cdot(\vec{E}\times\vec{H}^*) = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

Precisely from (9.21) we know that the time averaged power radiated in the direction θ is:
or per unit solid angle

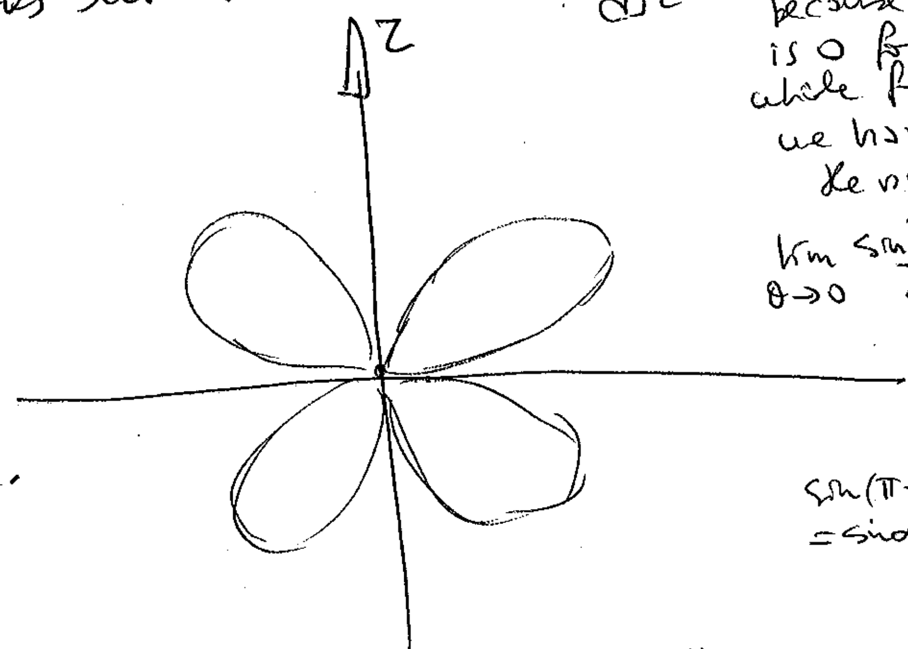
$$\frac{dP}{d\Omega} = \frac{1}{2} \operatorname{Re}\left[r^2 \hat{n}\cdot(\vec{E}\times H^*)\right]$$

the 1/2 and the * come from the time average

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2} \cdot \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}}.$$

(If we need the total power we need to integrate with $\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$.)

$\frac{dP}{d\Omega}$ looks like:



Because at Z=0, sin(kz) in original expression cancels, then dipolar contribution is gone.

$\frac{dP}{d\Omega} = 0$ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ because numerator is 0 for $\pi/2$ and $3\pi/2$, while for 0 and π we have to do carefully the ratio:

$$\lim_{\theta\to 0} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} = \frac{\sin^2[\pi(1-\theta^2/2)]}{\theta^2} = \frac{\sin^2(\pi\theta^2/2)}{\theta^2} = \frac{(\pi\theta^2/2)^2}{\theta^2} \to 0$$

$\sin(\pi-\alpha) = \sin\alpha$

Similar to a quadrupole pattern.

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