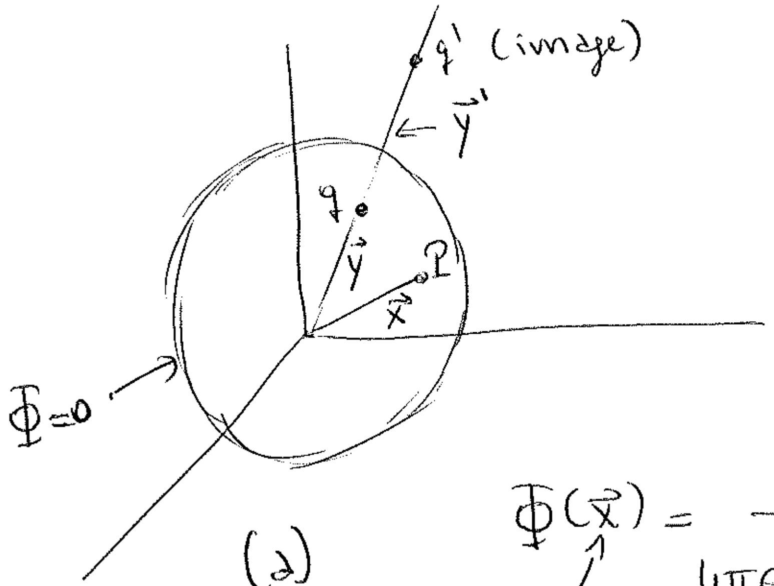
Problem 2.2 (just (a), (b))

In Sec. 2.2 the case of a charge q outside the sphere was studied. Now the charge q will be inside.



(a)

$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0 |\vec{x}-\vec{y}|} + \frac{q'}{4\pi\epsilon_0 |\vec{x}-\vec{y}'|}$$

inside sphere

$$\vec{x} = \cancel{x}\hat{x} = x\vec{n}$$
$$\vec{y} = y\vec{n}'$$

By symmetry q' is located along the same ray as $\vec{y}$: $\vec{y}' = y'\vec{n}'$

$$\Phi(\vec{x})\Big|_{\text{at surface}} = \frac{q/4\pi\epsilon_0}{a\left|\vec{n} - \frac{y}{a}\vec{n}'\right|} + \underbrace{\frac{q'/4\pi\epsilon_0}{a\left|\vec{n} - \frac{y'}{a}\vec{n}'\right|}}_{\frac{q'/4\pi\epsilon_0}{y'\left|\vec{n}' - \frac{a}{y'}\vec{n}\right|}}$$

as in (2.3)

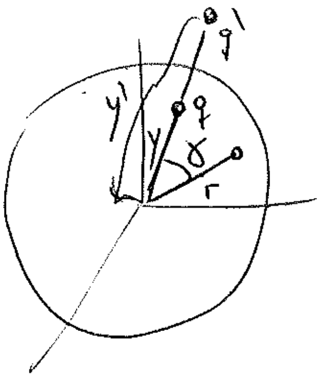
The conditions are then the same as before:

$$\frac{q}{a} = \frac{-q'}{y'} \quad ; \quad \frac{y}{a} = \frac{a}{y'}$$

or $\boxed{y' = \frac{a^2}{y}}$; $\boxed{q' = -\frac{q y'}{a}} = -\frac{q}{a}\frac{a^2}{y} = -\frac{q a}{y}$

If y is "small"
then y' is "large";
clearly outside
sphere.

$$\Phi(\vec{x}) = \frac{q/4\pi\epsilon_0}{\sqrt{r^2 + y^2 - 2ry\cos\gamma}} + \frac{\left(-q\frac{a}{y}\right)/4\pi\epsilon_0}{\sqrt{r^2 + \left(\frac{a^2}{y}\right)^2 - 2r\left(\frac{a^2}{y}\right)\cos\gamma}}$$



$$= \boxed{\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+y^2-2ry\cos\gamma}} - \frac{(a/y)}{\sqrt{r^2+\frac{a^4}{y^2} - \frac{2ra^2\cos\gamma}{y}}}\right]}$$

(b) To calculate the induced surface-charge density
we use:

$$\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a}$$

$$\frac{\partial \Phi}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{1}{2}\right)\left(r^2+y^2-2ry\cos\gamma\right)^{-3/2}(2r-2y\cos\gamma) - \left(\frac{a}{y}\right)\left(-\frac{1}{2}\right)\left(r^2+\frac{a^4}{y^2}-\frac{2ra^2\cos\gamma}{y}\right)^{-3/2}\left(2r-\frac{2a^2}{y}\cos\gamma\right)\right]$$

$\dfrac{\partial \Phi}{\partial r}$ evaluated at $r=a$ gives:

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{1}{2}\right)\frac{(2a-2y\cos\gamma)}{\underbrace{(a^2+y^2-2ay\cos\gamma)^{3/2}}_{y^{2\cdot 3/2}\left(\frac{a^2}{y^2}+1-\frac{2a}{y}\cos\gamma\right)^{3/2}}} + \frac{a}{2y}\frac{\left(2a-\frac{2a^2}{y}\cos\gamma\right)}{\underbrace{\left(a^2+\frac{a^4}{y^2}-2\frac{a^3}{y}\cos\gamma\right)^{3/2}}_{a^{2\cdot 3/2}\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma\right)^{3/2}}}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\underbrace{\left[\frac{1}{2y^3}\frac{(2y\cos\gamma-2a)}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}} + \frac{a}{2ya^3}\frac{\left(2a-\frac{2a^2}{y}\cos\gamma\right)}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}\right]}_{\frac{1}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}\underbrace{\left(\cancel{\frac{1}{y^2}\cos\gamma}-\frac{a}{y^3}+\frac{1}{ya}-\cancel{\frac{1}{y^2}\cos\gamma}\right)}_{\frac{1}{a^2}\frac{a}{y}\left(1-\frac{a^2}{y^2}\right)}}$$

$$\boxed{\epsilon_0\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a^2}\cdot\left(\frac{a}{y}\right)\cdot\frac{\overbrace{\left(\frac{a^2}{y^2}-1\right)}^{>0}}{\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{3/2}}} = \sigma$$

to make explicit the sign. The rest is (+).

Note that the unit vector $\vec{n}_{12}$ involved in the definition points inwards. Then there is a sign that we have to be careful with.

actually ~~up to the sign~~ it is the same expression (2.5) but with different sign. Since $y<a$ here while $y>a$ in (2.5), both are negative as they should.